Longo: Math 20C - Winter 2017
Lecture Notes

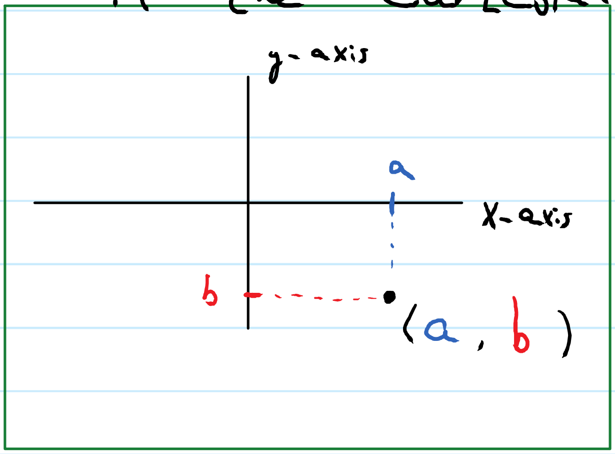
Date: January 9, 2017

Section:
- [ ] §1.1

Topics Covered:
- [ ] 2 - Space and 3 - Space / basic arithmetic

§1.1: Vectors and basic operations:

So far, you've dealt mostly with points (a, b) in the Cartesian plane:



From now on, we will denote the set of all points (a, b), where a and are real numbers by $\mathbb{R}^2$ ("2 dimensional real space", or simply "2 space").

In Calculus so far, you have been working with functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto y = f(x)$$

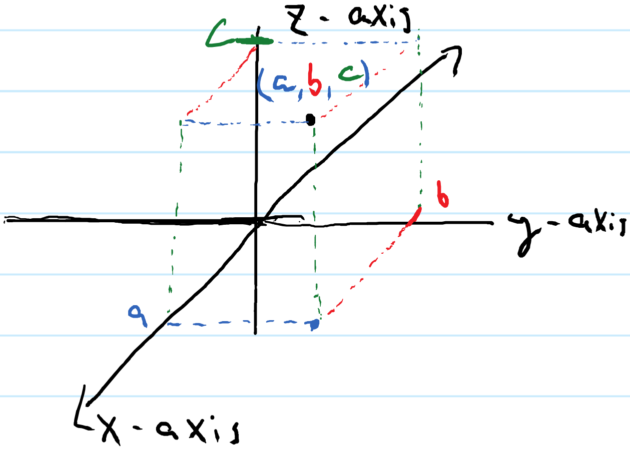
that take one real number 'x' as input, and return one real number 'y' as an output. In this class, we will try to do calculus on functions

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto z = f(x, y)$$

that take in a point '(x, y)' on the plane and spit out a real number 'z'.

Before we get there, we need to discuss geometry in 3 dimensional space in detail.

"3-space": Any triple (a, b, c) where a, b, c are real numbers represents a point in three dimensional space



In this picture, imagine the xy-plane is flat on the ground, and the z-axis is straight up out of the floor.

Remarks: ① The set of all such triples is denoted $\mathbb{R}^3$. ("3-space" or 3 dimentional real space).

② If (a, b, c) is a point in $\mathbb{R}^3$.

a is called the "x-component"

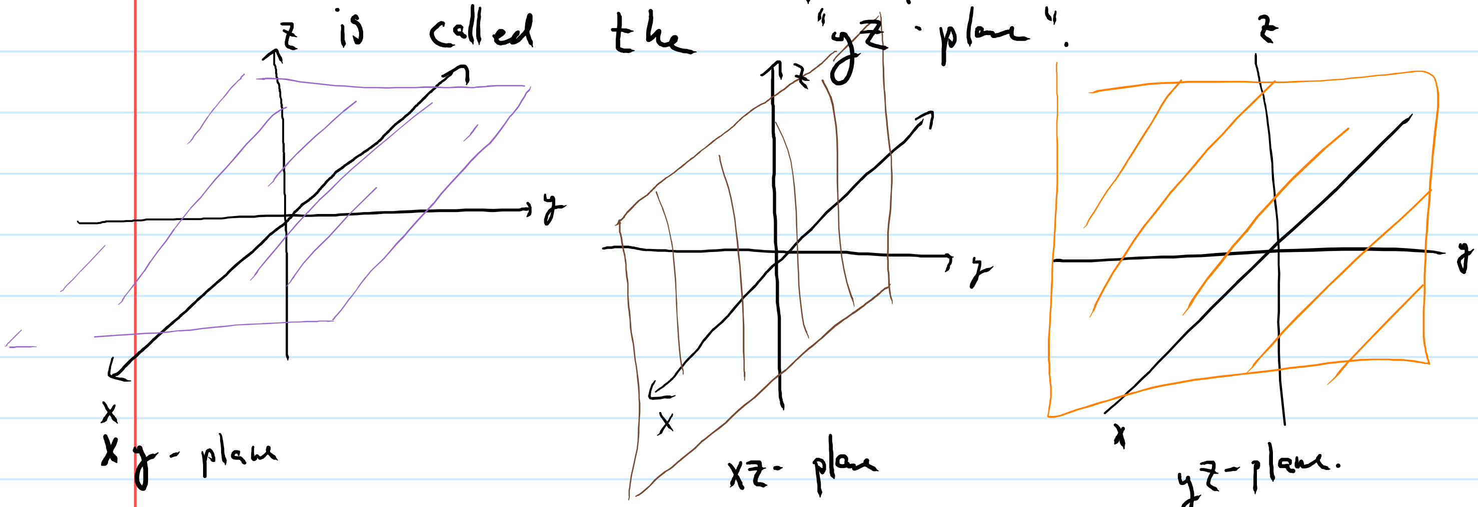
b is called the "y-component"

c is called the "z-component"

③ • The set of all points of the form $(a, b, 0)$ is called the "xy-plane"

• The set of all points of the form $(a, 0, c)$ is called the "xz-plane".

• The set of all points of the form $(0, b, c)$ is called the "yz-plane".



xy-plane xz-plane yz-plane.

Note: Planes are insanely hard to draw... sorry.

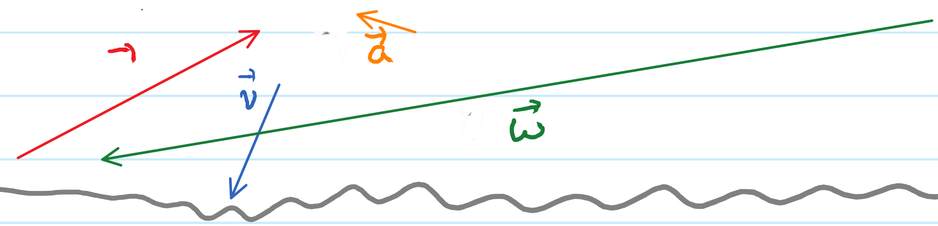
If we want to do Calculus on multivariable functions, we had better talk about how to do arithmetic using points in 2 or 3 space.

To do this, let's introduce the notion of

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$: For our purposes, a Vector in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is just a directed line segment in 2-space (resp. 3-space).

Since it is easier to draw in $\mathbb{R}^2$, let's start there.

Ex:

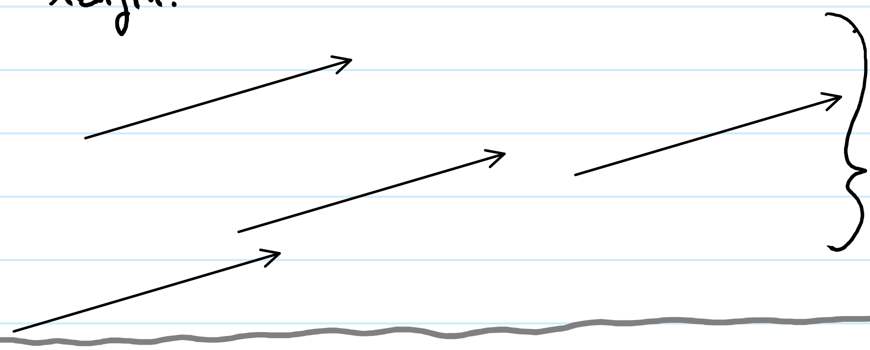


Remark: ① Vectors are usually written as a letter with an arrow above it: '$\vec{v}$'.

② A vector is determined by its length and direction. We don't really care where a vector starts or stops, what we are interested in is the relative change in position

With remark ② in mind, we call two vectors $\vec{u}$, $\vec{v}$ equivalent if they have the same direction and length.

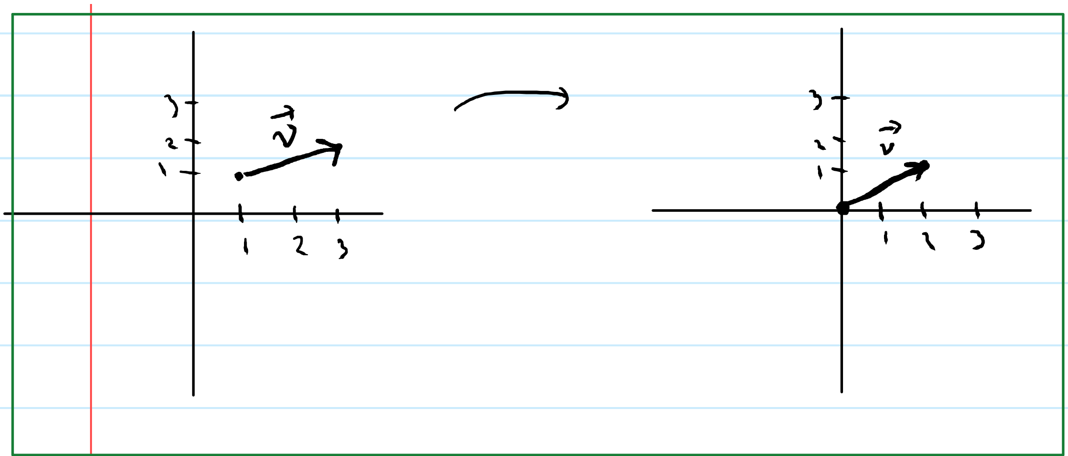
Ex:



all equivalent vectors.

How do we express vectors algebraically?

Given any vector, $\vec{v}$, $\mathbb{R}^2$ (or $\mathbb{R}^3$!), we can translate $\vec{v}$ so that its tail is at the origin.



Now the head of $\vec{v}$ stops at the point $(2,1)$. This means the vector goes $+2$ in the x-direction and $+1$ in the y-direction. By an abuse of notation, we say

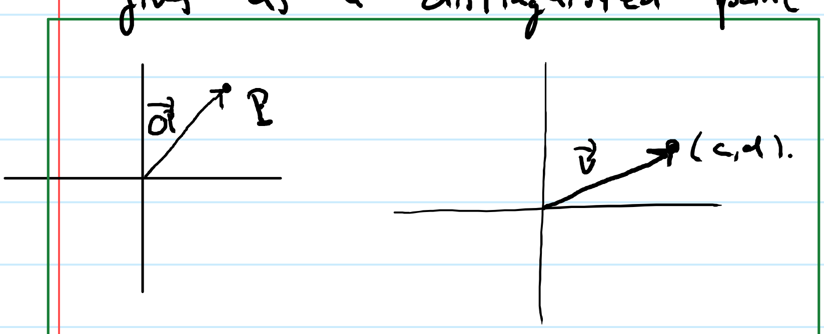
$$\boxed{\vec{v} = (2,1).}$$

Remark: Here, 2 is called the "x-component"
1 is called the "y-component"

Notice that this is the same notation we use for points in $\mathbb{R}^2$. We can do this because

Points and vectors starting at the origin are essentially the same thing

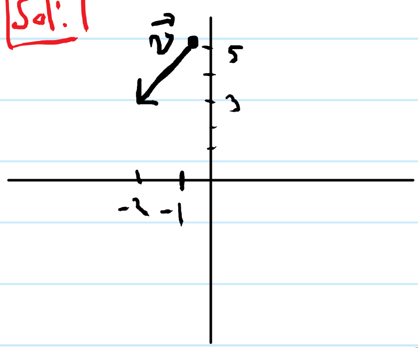
More Precisely, given a point $P=(a,b)$ on the plane, we can construct a vector starting at the origin by drawing a line from the origin to P. We call this vector $\vec{OP}$. Conversely if we have a vector $\vec{v} = (c,d)$ starting at the origin, its "head", (c,d) gives us a distinguished point in the plane.



We will often blur the notion of points and vectors into a single concept.

Example: Find the components of the vector that starts at $(-1,3)$ and ends at $(-2,5)$.

Sol:



Since $\vec{v}$ moves -1 in the x-direction and -2 in the y-direction.

$\vec{v} = (-1,-2)$

More generally the vector from $P=(x_1,y_1)$ to $Q=(x_2,y_2)$ is (x_2-x_1, y_2-y_1) and is denoted $\overrightarrow{PQ}$

Vector addition and scalar multiplication:

Remark: For us, "scalar" just means real number.

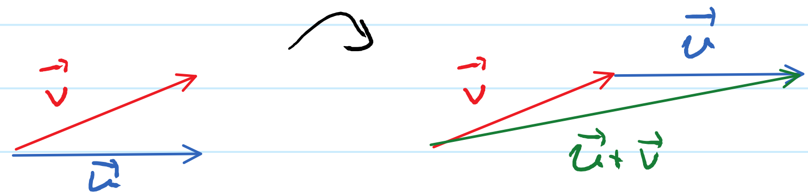
Given any two vectors $\vec{u}=(x_1,y_1)$, $\vec{v}=(x_2,y_2)$ and any real number k, we can add the vectors and multiply by the scalar k by doing the operations componentwise.

1. Vector addition: $\vec{u}+\vec{v} = (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$
2. scalar multiplication: $k\vec{u} = k(x_1,y_1) = (kx_1, ky_2)$

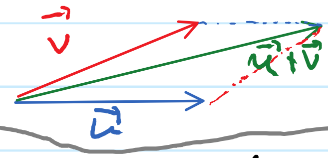
Rmk: Vector addition and scalar multiplication have nice properties such as commutativity, associativity, and distribution. (See book for details)

What does this mean geometrically?

① Given two vectors, $\vec{u}, \vec{v}$, draw them so that they start at a common point.

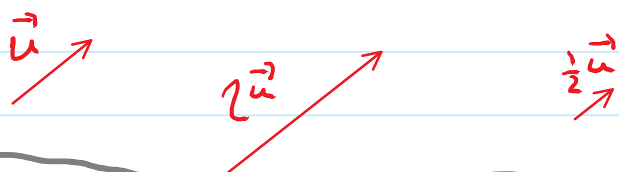


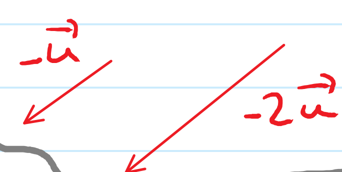
move $\vec{u}$ so that its tail is on the head of $\vec{v}$. $\vec{u}+\vec{v}$ is the vector that starts at the base of $\vec{v}$ and ends at the head of $\vec{u}$. Equivalently, use the parallelogram rule:



② If $k>0$, then $k\vec{u}$ is just what you get if you stretch $\vec{u}$ by a factor of k. If $k<0$, the resulting vector points the opposite direction, as well as being stretched.

Ex:

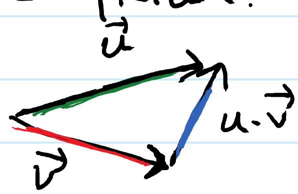




What is $\vec{u}-\vec{v}$ geometrically?

Notice the vector $\vec{u}-\vec{v}$ is what you add to $\vec{v}$ in order to get $\vec{u}$: $\boxed{\vec{v}+(\vec{u}-\vec{v})=\vec{u}}$

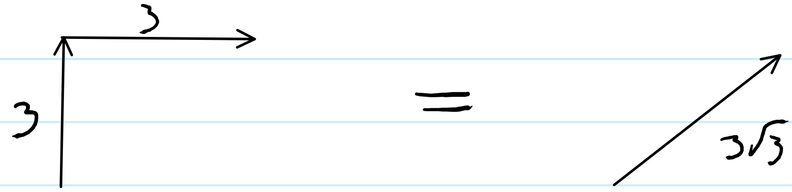
So we have the picture:



So $\vec{u}-\vec{v}$ goes from the head of $\vec{v}$ to the head of $\vec{u}$.

Remark: You should think of vector addition, $\vec{u}+\vec{v}$ as moving along $\vec{u}$ and then moving along $\vec{v}$.

For example: "Go North 3 units" + "Go East 3 units" is the same as "Go North-East $3\sqrt{2}$ units".



Next time: ① Length of vectors / normalization
② lines.
③ inner product.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 11, 2017

Section:
- ☐ § 1.1 (cont.)
- ☐ § 1.2

Topics Covered:
- ☐ Special vectors
- ☐ Equations of lines in 3-space
- ☐ Dot product of vectors (also called inner product)

From last time:

A **vector** in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is a directed line segment in 2-space (resp. 3-space). Two vectors are equivalent if they are translates of each other. This means we only really care about the **length** and **direction** of a vector.

Special Vectors:

The following vectors are especially important:

① The **zero vector**, $\vec{0}$, is the vector that starts and ends at the origin. $\vec{0}$ is the only vector of length zero. In $\mathbb{R}^2$, $\vec{0} = (0,0)$ in $\mathbb{R}^3$, $\vec{0} = (0,0,0)$.

Remark: If $\vec{v}$ is any vector, $\boxed{\begin{gathered}\vec{0} + \vec{v} = \vec{v} \\ \vec{v} + \vec{0} = \vec{v}.\end{gathered}}$

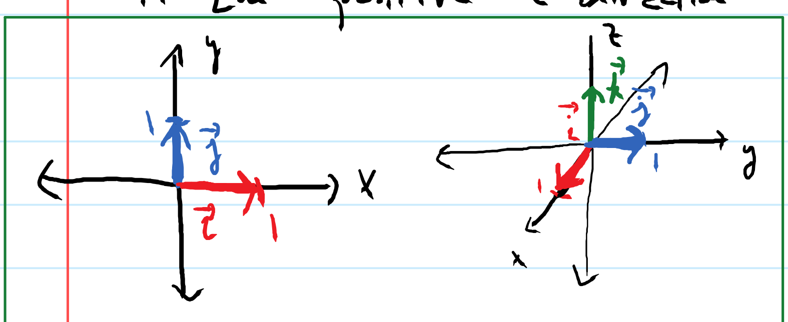
② The vector $\vec{i}$ is the vector of length 1 that points in the positive x-direction.

$\vec{i} = (1,0)$ in $\mathbb{R}^2$ and $\vec{i} = (1,0,0)$ in $\mathbb{R}^3$.

③ The vector $\vec{j}$ is the vector of length 1 that points in the positive y-direction.

$\vec{j} = (0,1)$ in $\mathbb{R}^2$ and $\vec{j} = (0,1,0)$ in $\mathbb{R}^3$.

④ The vector $\vec{k}$ is the vector of length 1 that points in the positive z-direction $\vec{k} = (0,0,1)$.



$\vec{i}, \vec{j}, \vec{k}$ are important because any vector can be expressed in terms of $\vec{i}, \vec{j}, \vec{k}$.

Ex: ① $(-1,3) = (-1,0) + (0,3) = -1(1,0) + 3(0,1)$

$= -\vec{i} + 3\vec{j}$

② $(6,-2,4) = 6(1,0,0) - 2(0,1,0) + 4(0,0,1) = 6\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are called the standard basis vectors.

Equations of lines in 3-space:

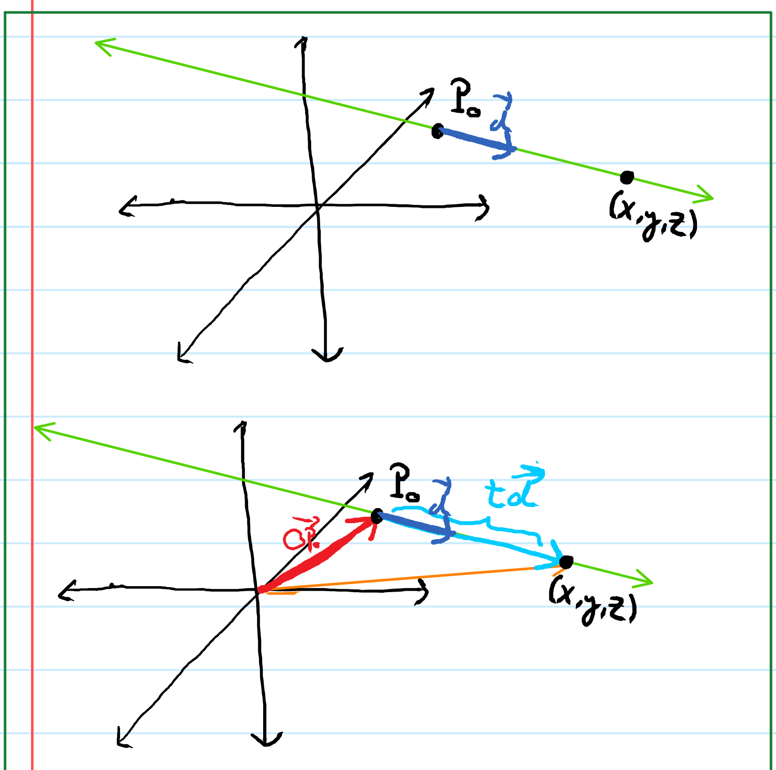
You know that in $\mathbb{R}^2$, you can find the equation of a line if you know:
1. a point, (x_0, y_0), on the line, and
2. The slope, m, (direction of the line).

Then we can use the point-slope form:

$$(y - y_1) = m(x - x_1).$$

We do something similar in 3-space.

Suppose you know a point $P_0 = (x_0, y_0, z_0)$ on a line, and a vector $\vec{d} = (a, b, c)$ that is parallel to the line.



If (x,y,z) is any other point on the line, we can write $\boxed{(x,y,z) = \vec{OP_0} + t\vec{d}}$ for some real number t.

So any point on the line has the form $\vec{OP_0} + t\vec{d}$ for some t. We say the line can be described parametrically with parameter t, by

$$\boxed{\begin{aligned} \vec{\ell}(t) &= \vec{OP_0} + t\vec{d} \\ &= (x_0, y_0, z_0) + t(a, b, c) \\ &= (x_0 + ta, y_0 + tb, z_0 + tc) \end{aligned}}$$

vector form

Equivalently, the line is described by the parametric equations:

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

Examples: ① Find the line passing through $(0,9,-1)$ that is parallel to the vector $(6,-2,-1)$.

② Find the eqn of the line passing through $(0,9,-1)$ and $(-1,-1,1)$.

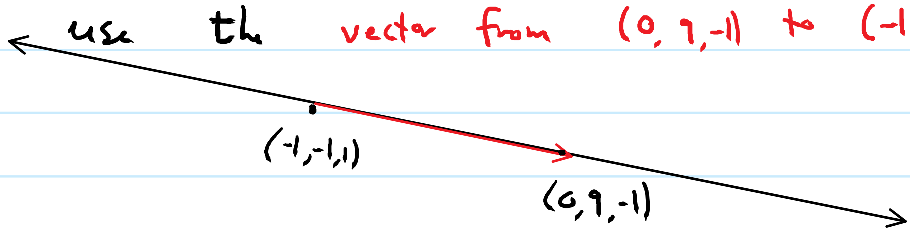
③ Where do the lines $\vec{l}_1(t) = (1,0,0) + t(2,-1,1)$, $\vec{l}_2(t) = (3,-3,-1) + t(0,1,1)$ intersect?

Sol: ① Here, $P_0 = (0,9,-1)$, $\vec{d} = (6,-2,-1)$ So the line is

$$\boxed{\vec{l}(t) = (0,9,-1) + t(6,-2,-1)}$$

② Recall, we just need any point on the line, and a direction vector, $\vec{d}$.

Let's take $P_0 = (-1,-1,1)$. For $\vec{d}$ we can use the vector from $(0,9,-1)$ to $(-1,-1,1)$.



So take $\vec{d} = (0,9,-1) - (-1,-1,1) = (0+1, 9+1, -1-1) = (1,10,-2)$

So we get $\boxed{\vec{l}(t) = (-1,-1,1) + t(1,10,-2).}$

③ If the two lines intersect, then the x,y,z components must all be equal. So we set up the equations:

$$(1,0,1) + t(2,-1,1) = (4,-3,0) + s(0,1,1)$$

Rmk: We must change one of the t's to a different variable because they need not intersect at the same "input value".

So we have: ① $1 + 2t = 4$

② $-t = -3 + s$

③ $1 + t = s$

From eqn ③ $s = 1+t$, plug this into eqn ②

$$-t = -3 + (1+t)$$

$$\Rightarrow -2t = -3$$

$$\Rightarrow \boxed{t = \tfrac{3}{2}}$$

Plug this back into (3):

$$\boxed{s = 1 + \frac{3}{2} = \frac{5}{2}}$$

So the y and z coords are equal when $t = \frac{3}{2}$ and $s = \frac{5}{2}$. Let's plug this back into eqn (1) to see if the x-coords are equal:

$$1 + 2\left(\tfrac{3}{2}\right) \stackrel{?}{=} 4 \quad \checkmark.$$

$\therefore$ They intersect when say $t = \frac{3}{2}$. (or $s = \frac{5}{2}$)

we get $\left(1 + 2\left(\frac{3}{2}\right), -\frac{3}{2}, 1 + \frac{3}{2}\right) = \boxed{\left(4, -\frac{3}{2}, \frac{5}{2}\right)}$

§1.2: The Inner Product:

So far, we discussed how to multiply a vector with a scalar (real number). There are two useful ways to multiply two vectors (neither are what you probably expect). The first way is called the "dot product", or, "inner product".

Def: Let $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$ be two vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is defined by

$$\boxed{\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

The dot product of two vectors in $\mathbb{R}^2$ is the same, but you don't include the z-components

Remarks: (1) The dot product of two vectors is a number!

(2) The dot product has nice properties:

For vectors $\vec{u}, \vec{v}, \vec{w}$, and a scalar k,

(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutativity)

(ii) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

(iii) $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$ (distributivity)

!Warning! DO NOT FORGET THE DOT!! "$\vec{u}\vec{v}$" has NO meaning.

Example: ① Let $\vec{u} = (0,3,-1)$, $\vec{v} = (5,5,0)$.

$$\vec{u}\cdot\vec{v} = (0)(5) + (3)(5) + (-1)(0) = \boxed{15}$$

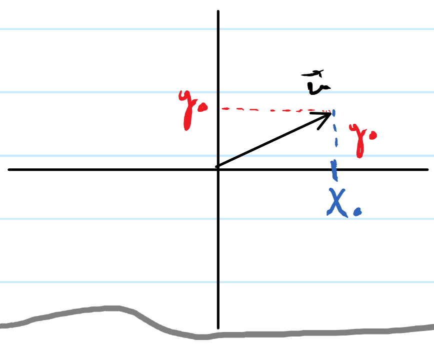
② Let $\vec{a} = (9, 5)$, $\vec{b} = (-5, 9)$

$$\vec{a}\cdot\vec{b} = (9)(5) + (-5)(9) = 45 - 45 = \boxed{0}$$

The dot product and length:

Q? If $\vec{u} = (x_0, y_0)$ is a vector in $\mathbb{R}^2$, what is the length of $\vec{u}$?

Notation: The length of a vector $\vec{u}$ is denoted $\|\vec{u}\|$. $\|\vec{u}\|$ is also called the norm or magnitude of $\vec{u}$



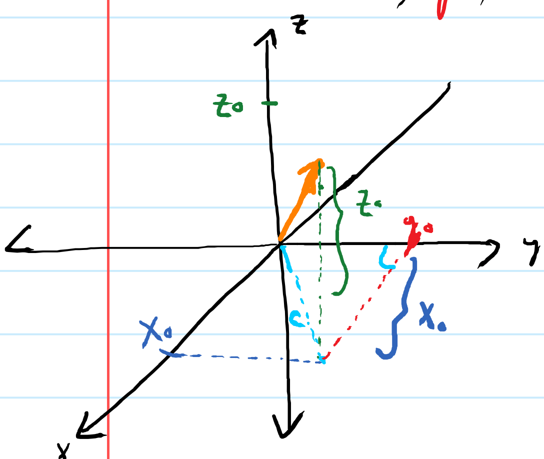
Using Pythagorean thm:

$$\|\vec{u}\|^2 = x_0^2 + y_0^2 \Rightarrow$$

$$\boxed{\|\vec{u}\| = \sqrt{x_0^2 + y_0^2}}$$

What about in $\mathbb{R}^3$?

Let $\vec{v} = (x_0, y_0, z_0)$.



We use Pythagorean Thm twice. In this picture, we look at the right triangle in the xy-plane. The hypotenuse, , has length

$$c = \sqrt{x_0^2 + y_0^2}$$

Now using the Pythagorean thm on the orange, green, l. blue right triangle, we get.

$$\|\vec{v}\|^2 = c^2 + z_0^2$$

$$\boxed{\|\vec{v}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}}$$

Note that if we calculate:

$$\vec{v}\cdot\vec{v} = (x_0, y_0, z_0)\cdot(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2.$$

Therefore. $\boxed{\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}}$ (Important!)

Example: Find the magnitude of $\vec{u} = (1,1,2)$, $\vec{v} = (6,-7)$

Sol: $\|\vec{u}\| = \sqrt{(1)^2+(1)^2+(2)^2} = \sqrt{1+1+4} = \boxed{\sqrt{6}}$

$\|\vec{v}\| = \sqrt{(6)^2+(-7)^2} = \sqrt{36+49} = \boxed{\sqrt{85}}$

Next time: ① Unit vectors / Normalization

② Geometry of the inner product & Orthogonality.

③ Orthogonal Projections.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 13, 2017

Section:

- [ ] §1.2 (cont.)

Topics Covered:

- [ ] Unit vectors and normalization
- [ ] Geometry of the inner product and orthogonality
- [ ] Orthogonal projections

§1.2 (cont.):

Recall from last time: If $\vec{u} = (x_0, y_0, z_0)$ is a vector in $\mathbb{R}^3$. $\boxed{\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{x_0^2 + y_0^2 + z_0^2}}$

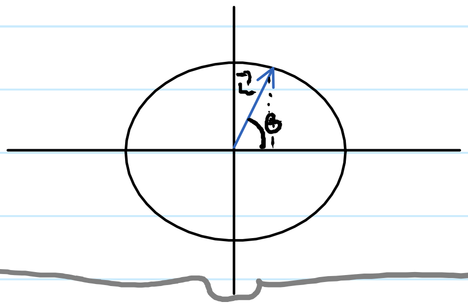
For vectors in $\mathbb{R}^2$, just ignore the z-component.

Unit vectors: A vector $\vec{u}$ is called a unit vector iff $\|\vec{u}\| = 1$.

> **Remark:** The word "unit" comes from the latin word 'unum', meaning 1.

Example: (Unit Vectors in $\mathbb{R}^2$:). Suppose $\vec{u} = (x_0, y_0)$ is a unit vector. Then $x_0^2 + y_0^2 = 1$.

Therefore, if we put the tail of $\vec{u}$ at the origin, then the head is on the unit circle.



If θ is the angle between $\vec{u}$ and the positive real axis, then $\boxed{\vec{u} = (\cos\theta, \sin\theta)}$.

Let $\vec{u}$ be any vector. If we multiply $\vec{u}$ by the scalar $\frac{1}{\|\vec{u}\|}$, it "shrinks" $\vec{u}$ by a factor of $\|\vec{u}\|$. The resulting vector, $\boxed{\vec{e}_{\vec{u}} = \frac{1}{\|\vec{u}\|}\vec{u}}$, has length 1. This process is called normalization.

Example: Normalize $\vec{u} = (-6, 1, 9)$.

Calculate: $\|\vec{u}\| = \sqrt{(-6)^2 + (1)^2 + (9)^2} = \sqrt{118}$

so
$$\boxed{\begin{aligned} e_{\vec{u}} &= \frac{1}{\sqrt{118}}(-6, 1, 9) \\ &= \left(\frac{-6}{\sqrt{118}}, \frac{1}{\sqrt{118}}, \frac{9}{\sqrt{118}}\right) \end{aligned}}$$

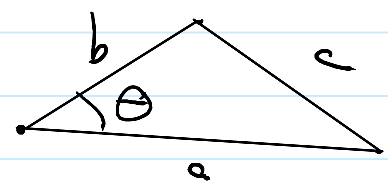
Geometry of the inner product:

We haven't yet discussed why we use the dot product. The summary is:

The dot product tells you about the (smallest) angle between two vectors.

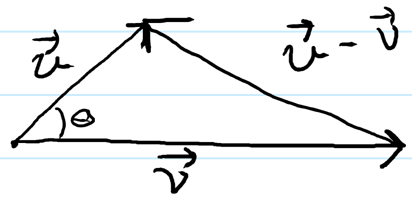
Let's see how. First we need the

Law of Cosines: In the triangle:



we have $a^2+b^2=c^2+2ab\cos\theta$

Let $\vec{u}, \vec{v}$ be two vectors. Draw them so that they start at a common point, and draw the difference vector, $\vec{u}-\vec{v}$. Let θ be the smallest angle between $\vec{u},\vec{v}$.



By the law of cosines:

(∗) $$\|\vec{u}\|^2+\|\vec{v}\|^2=\|\vec{u}-\vec{v}\|^2+2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

Since
$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot(\vec{u}-\vec{v})-\vec{v}\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot\vec{u}-\underbrace{\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}}+\vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2-2(\vec{u}\cdot\vec{v})+\|\vec{v}\|^2.\end{aligned}$$

So eqn (*) becomes

$$\cancel{\|\vec{u}\|^2}+\cancel{\|\vec{v}\|^2} = \cancel{\|\vec{u}\|^2} - 2(\vec{u}\cdot\vec{v}) + \cancel{\|\vec{v}\|^2} + 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

$$\Rightarrow \quad 0 = -2(\vec{u}\cdot\vec{v}) + 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Rightarrow \quad \boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta}$$

Important!

<u>Example</u>: What is the smallest angle, θ, between $\vec{u} = (-1,0,1)$, $\vec{v} = (7,-13,4)$.

Sol: The above equation implies:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right).$$

Calculate:

$$\boxed{\vec{u}\cdot\vec{v} = (-1)(7) + (0)(-13) + (1)(4) = -3}$$

$$\boxed{\|\vec{u}\| = \sqrt{(-1)^2+(0)^2+(1)^2} = \sqrt{2}}$$

$$\boxed{\|\vec{v}\| = \sqrt{(7)^2+(-13)^2+4^2} = \sqrt{49+169+16} = \sqrt{234}}$$

So $\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{2}\cdot\sqrt{236}}\right).}$

<u>Remarks</u>: Since $\|\vec{u}\|, \|\vec{v}\| > 0$,

1. $\vec{u}\cdot\vec{v} > 0 \Leftrightarrow \cos\theta > 0 \Leftrightarrow 0 \leq \theta < \frac{\pi}{2}$ (θ is acute)
2. $\vec{u}\cdot\vec{v} < 0 \Leftrightarrow \cos\theta < 0 \Leftrightarrow \frac{\pi}{2} < \theta \leq \pi$ (θ is obtuse)

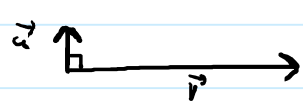
and most importantly,

3. $\vec{u}\cdot\vec{v} = 0$ iff $0 = \|\vec{u}\|\|\vec{v}\|\cos\theta$

   $\vec{u}\cdot\vec{v} = 0$ iff $\cos\theta = 0$

   $\boxed{\vec{u}\cdot\vec{v} = 0 \text{ iff } \theta = \frac{\pi}{2} \; (=90^\circ).}$

Important



Since the dot product is so easy to calculate this is a very quick test to see if two vectors meet at a right angle.

Terminology: Two vectors $\vec{u}, \vec{v}$ are said to be orthogonal or normal or perpendicular if they meet at a right angle (iff $\vec{u}\cdot\vec{v} \overset{?}{=} 0$). We denote this by $\vec{u} \perp \vec{v}$.

Example: Decide if the angle between $\vec{u} = (-1, 1, 7)$, $\vec{v} = (-3, 10, 5)$ is obtuse, accute, or right.

Sol: Use the dot product!

$$\vec{u}\cdot\vec{v} = (-1)(-3) + (1)(10) + (7)(5) > 0$$

so the angle between $\boxed{\vec{u}, \vec{v} \text{ is acute.}}$

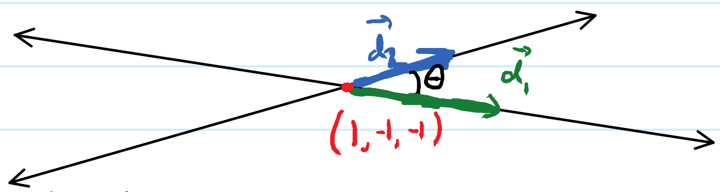
Example: Prove that the lines

$$\ell_1(t) = (1,-1,-1) + t(-5,2,2)$$
$$\ell_2(t) = (1,-1,-1) + t(6,9,6)$$

intersect at a right angle.

Sol: We can tell by inspection that the point $(1,-1,-1)$ is on both lines, so they definitely intersect.

Notice that to find the angle between ℓ_1, ℓ_2, we need to find the angle between their direction vectors $\vec{d_1} = (-5,2,2)$, $d_2 = (6,9,6)$.

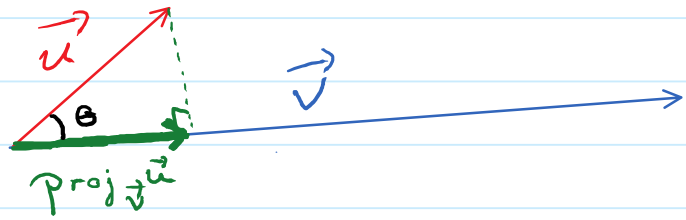


Note: $\vec{d_1}\cdot\vec{d_2} = (-5)(6) + (2)(9) + (2)(6) = -30 + 18 + 12 = 0$.

Therefore $\ell_1 \perp \ell_2$.

Orthogonal Projections:

Start with two vectors $\vec{u}, \vec{v}$.



Imagine you shine a flashlight straight down onto $\vec{v}$. The shadow that $\vec{u}$ casts onto $\vec{v}$ is called the projection of $\vec{u}$ along (or onto) $\vec{v}$. It is denoted $\text{proj}_{\vec{v}}\vec{u}$.

To determine $\text{proj}_{\vec{v}}\vec{u}$, we just need its direction and magnitude:

Using basic trigonometry, we see $\cos(\theta) = \dfrac{\|\text{proj}_{\vec{v}}\vec{u}\|}{\|\vec{u}\|}$

$$\Rightarrow \boxed{\|\text{proj}_{\vec{v}}\vec{u}\| = \|\vec{u}\|\cos\theta = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|}.}$$

Secondly, $\text{proj}_{\vec{v}}\vec{u}$ points in the direction of $\vec{v}$. So the unit vector that points in the direction of $\text{proj}_{\vec{v}}\vec{u}$ is $\pm\dfrac{\vec{v}}{\|\vec{v}\|}$.

If we are careful with the $\pm$ sign we see

$$\boxed{\begin{aligned}\text{proj}_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}\\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}\\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\end{aligned}}$$

Example: Find the projection of $\vec{u} = \langle 9, -4 \rangle$ onto $\vec{v} = \langle 2, 2 \rangle$

Sol:

$$\vec{u} \cdot \vec{u} = (9)(2) + (-4)(2)$$
$$= 18 - 8 = 10$$
$$\vec{v} \cdot \vec{v} = (2)(2) + (2)(2) = 8$$

$$\boxed{\begin{aligned} \text{proj}_{\vec{v}}\, \vec{u} &= \left(\frac{\vec{u} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{10}{8} \langle 2, 2 \rangle \\ &= \frac{5}{4} \langle 2, 2 \rangle \\ &= \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle \end{aligned}}$$

Longo: Math 20C - Winter 2017
Lecture Note

Date: January 18,2017

Section:
- [ ] §1.2 (cont.)
- [ ] §1.3

Topics Covered:
- [ ] Orthogonal projections
- [ ] Determinants, area and volume

Since the dot product is so easy to calculate this is a very quick test to see if two vectors meet at a right angle.

Terminology: Two vectors $\vec{u}, \vec{v}$ are said to be **orthogonal** or **normal** or **perpendicular** if they meet at a right angle (iff $\vec{u}\cdot\vec{v} \overset{?}{=} 0$). We denote this by $\vec{u} \perp \vec{v}$.

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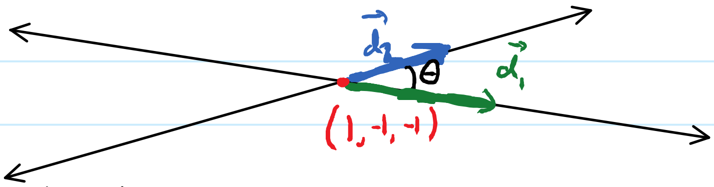

Note: $\vec{d_1}\cdot\vec{d_2} = (-5)(6) + (2)(9) + (2)(6) = -30+18+12 = 0.$

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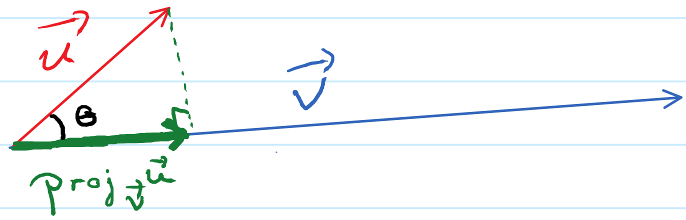

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§ 1.3: The cross Product:

The immediate goal is to define the second way we can "multiply" two vectors. We will eventually use it to come up with equations of planes. First, we revew determinants.

Let $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ be a 2×2 matrix. The determinant of A, denoted $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ or $\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, is defined by:

$$\boxed{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1}$$

Example: $\begin{vmatrix} 4 & 7 \\ -3 & 10 \end{vmatrix} = (4)(10) - (7)(-3) = 40 + 21 = \boxed{61}$

Rmk: We will see shortly that the determinant of a matrix carries information about area/volume.

Q! What about 3×3 matrices?

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. To find the determinant, we "expand along the first row:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

To get this, circle a_1 and cross out its row and column. The first term is $a_1 \times$ (the determinant of the 2×2 matrix that is not crossed out). Then move onto a_2 and do the same thing, except now there is a negative sign. Then move onto a_3 but with a + sign.

Example: $\begin{vmatrix} 1 & -1 & 2 \\ -3 & 0 & 4 \\ 9 & -5 & -5 \end{vmatrix} = 1 \begin{vmatrix} 0 & 4 \\ -5 & -5 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 4 \\ 9 & -5 \end{vmatrix} + 2 \begin{vmatrix} -3 & 0 \\ 9 & -5 \end{vmatrix}$

$$= 1\big((0)(-5) - (4)(-5)\big) - (-1)\big((-3)(-5) - (4)(9)\big) + 2\big((-3)(-5) - (0)(9)\big)$$
$$= 1(20) + 1(15-36) + 2(15)$$
$$= 20 - 21 + 30$$
$$= \boxed{29}$$

The Cross Product:

We use the determinant to define the "cross product" of two vectors in $\mathbb{R}^3$.

Def: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The cross product of $\vec{u}$ and

$\vec{v}$, denoted by $\vec{u}\times\vec{v}$, is defined by:

$$\boxed{\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}}$$

<u>Example</u>: Let $\vec{u} = (5,5,0)$, $\vec{v} = (-1,1,1)$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 5 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 5 & 0 \\ -1 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 5 & 5 \\ -1 & 1 \end{vmatrix}\vec{k}$$

$$= 5\vec{i} - 5\vec{j} + 10\vec{k}$$

$$= (5, -5, 10)$$

<u>First Properties</u>: (1) Anticommutativity: $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

(2) $(k\vec{u})\times\vec{v} = k(\vec{u}\times\vec{v}) = \vec{u}\times(k\vec{v})$.

(3) $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$

$\vec{w}\times(\vec{u}+\vec{v}) = (\vec{w}\times\vec{u}) + (\vec{w}\times\vec{v})$

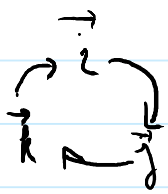
<u>Rmk</u>: (1) Prop. (1) says the order that you put $\vec{u}, \vec{v}$ into the matrix matters, in $\vec{u}\times\vec{v}$, $\vec{u}$ goes in row 2, $\vec{v}$ goes in row 3.

(2) These are proved using basic calculations. Props. (1), (2) also come from basic determinant properties.

<u>Fact</u>: Easy calculations show:

$$\boxed{\begin{aligned} \vec{i}\times\vec{j} &= \vec{k} \\ \vec{j}\times\vec{k} &= \vec{i} \\ \vec{k}\times\vec{i} &= \vec{j} \end{aligned}}$$

An easy way to remember this is the diagram:



Next time we discuss the geometry of the cross product.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 20, 2017

Section:

- [ ] §1.3 (cont.)

Topics Covered:

- [ ] The Triple Product the geometry of the cross product, and equations of planes.

Geometry of the Cross Product:

Last time we saw that if $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, the cross product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Q? What are the magnitude and direction of $\vec{u} \times \vec{v}$?

Before we get there, let's discuss the "triple product" of vectors: Let $\vec{w} = (w_1, w_2, w_3)$, then

$$\begin{aligned}(\vec{u} \times \vec{v}) \cdot \vec{w} &= \left(\begin{vmatrix} u_2 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_3 \end{vmatrix} \vec{k} \right) \cdot \left(w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k} \right) \\ &= \begin{vmatrix} u_2 & u_3 \\ v_1 & v_3 \end{vmatrix} w_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} w_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_3 \end{vmatrix} w_3 \\ &= \boxed{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}\end{aligned}$$

A simple calculation tells us:

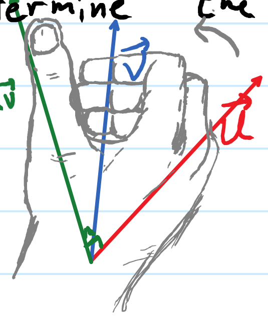
$$(\vec{u} \times \vec{v}) \cdot \vec{u} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0.$$ Similarly

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = 0.$$

This tells us $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

This narrows the direction of $\vec{u} \times \vec{v}$ down to 2 choices. to determine the direction, use the "right hand rule".

This is really the best I can do



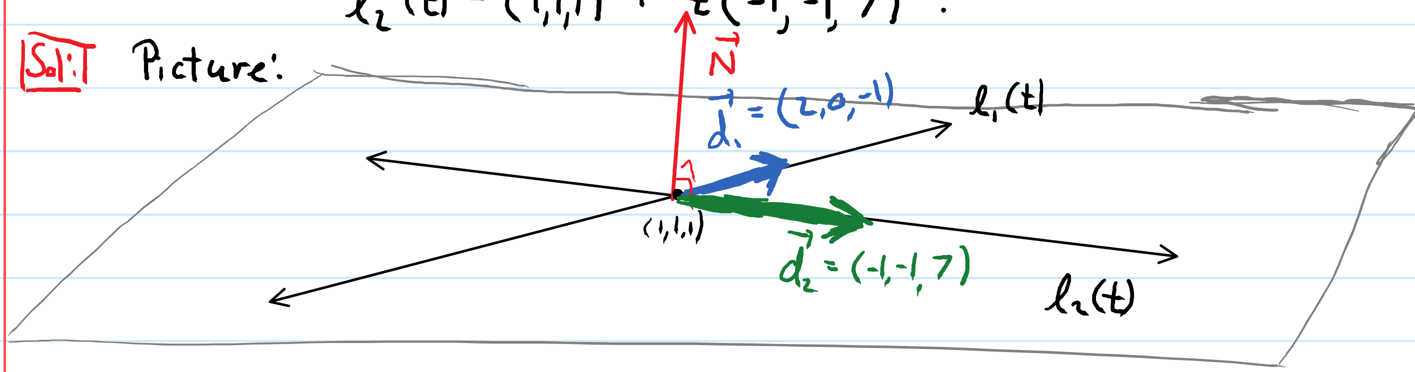
Point your fingers in the direction of $\vec{u}$, then curl them towards $\vec{v}$. Your thumb points at $\vec{u} \times \vec{v}$.

$\vec{N}$

Example: Find a vector $\vec{N}$ that is orthogonal to the plane that contains the lines

$$\vec{l}_1(t) = (1,1,1) + t(2,0,-1)$$
$$\vec{l}_2(t) = (1,1,1) + t(-1,-1,7).$$

Sol: Picture:



As we see in the picture, $\vec{N}$ is normal to the plane containing $\vec{l}_1(t)$, $\vec{l}_2(t)$ iff $\vec{N} \perp \vec{d}_1$ and $\vec{N} \perp \vec{d}_2$.
where $\vec{d}_1 = (2,0,-1)$ is the direction vector for $\vec{l}_1(t)$
and $\vec{d}_2 = (-1,-1,7)$ is the direction vector for $\vec{l}_2(t)$.
So for $\vec{N}$, we can use

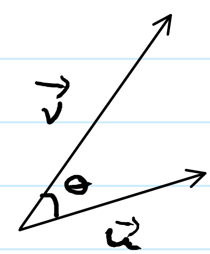
$$\vec{N} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -1 & -1 & 7 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -1 & 7 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ -1 & 7 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} \vec{k}$$

$$\boxed{= (1, -13, -2)}$$

What is $\|\vec{u} \times \vec{v}\|$?

It is not terribly hard to prove:

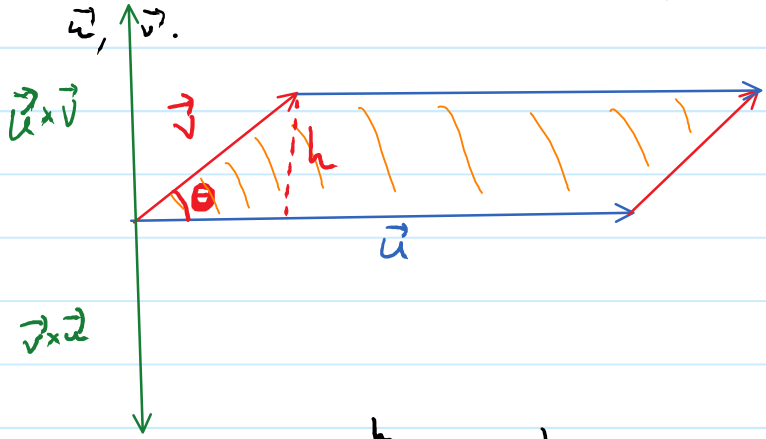
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$$



where θ is the (smallest) angle between $\vec{u}$, $\vec{v}$.
(See proof on pg. 36 of book. It's not worth doing in class.).

What does this mean?

Let's look at the area of the parallelogram spanned by $\vec{u}, \vec{v}$.



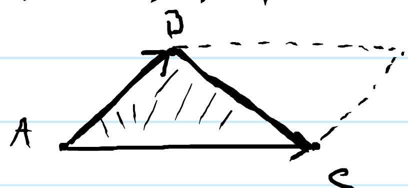
basic trig tells us $\sin(\theta) = \frac{h}{\|\vec{v}\|} \Rightarrow h = \|\vec{v}\| \sin\theta$.

Therefore: $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta =$ Area of the parallelogram.

Example: Find the area of the triangle w/ vertices

$$A = (-1, 3, 4), \quad B = (0, 5, 6), \quad C = (-7, -7, 0).$$

Sol:



The area of ΔABC is $\frac{1}{2}$(parallelogram) spanned by $\vec{AB}, \vec{AC}$. So Area$(\Delta ABC) = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$.

$$\vec{AB} = \vec{OB} - \vec{OA} = (0-(-1), 5-3, 6-4) = (1, 2, 2)$$
$$\vec{AC} = \vec{OC} - \vec{OA} = (-7-(-1), -7-3, 0-4) = (-6, -10, -4).$$

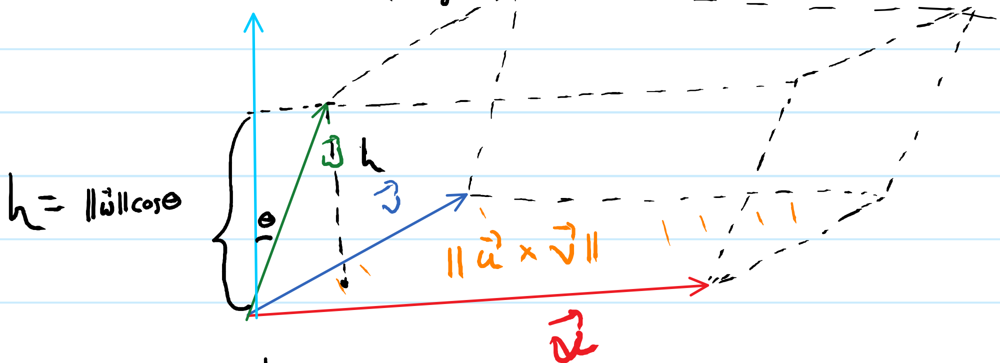
$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ -6 & -10 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ -10 & -4 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ -6 & -4 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ -6 & -10 \end{vmatrix}\vec{k}$$
$$= (12, -8, 2)$$

So
$$\frac{1}{2}\|\vec{AB}\times\vec{AC}\| = \frac{1}{2}\left(\sqrt{12^2 + (-8)^2 + 2^2}\right)$$
$$= \frac{1}{2}\sqrt{144 + 64 + 4}$$
$$= \boxed{\frac{1}{2}\sqrt{212}}$$

Geometry of the triple product: $(\vec{u}\times\vec{v})\cdot\vec{w}$.

We know: $(\vec{u}\times\vec{v})\cdot\vec{w} = \|\vec{u}\times\vec{v}\|\,\|\vec{w}\|\cos\theta$ where θ is the angle between $\vec{u}\times\vec{v}$ and $\vec{w}$. On the other hand, $\|\vec{u}\times\vec{v}\|$ is the area of the parallelogram spanned by $\vec{u}, \vec{v}$, and $\|\vec{w}\|\cos\theta = \|\text{proj}_{\vec{u}\times\vec{v}}\vec{w}\|$ (see notes from 1-18).

picture:



So $|(\vec{u}\times\vec{v})\cdot\vec{w}| =$ volume of the "parallelopiped" spanned by $\vec{u}, \vec{v}, \vec{w}$.

Example: Find the volume of the parallelopiped spanned by $\vec{u}\,(1,1,1)$, $\vec{v}=(-2,0,12)$, $\vec{w}=(8,0,0)$

Sol: Volume $= |(\vec{u}\times\vec{v})\cdot\vec{w}| = \left|\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 12 \\ 8 & 0 & 0 \end{vmatrix}\right|$

$= |(8)(12)| = \boxed{96}$

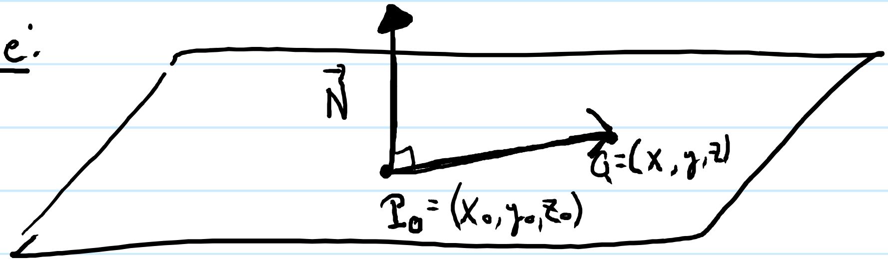
Equations of planes:

To find the eqn of a plane, we need:

1. A point, $P_0 = (x_0, y_0, z_0)$, on the plane.
2. A vector, $\vec{N} = (a, b, c)$, that is normal to the plane. } $\vec{N}$ is called the "normal vector"

Compare this to what we need for the eqn of a line! We can think of $\vec{N}$ as a vector that orients the plane (direction).

Picture:



If Q is any point on the plane, then $\overrightarrow{P_0Q}$ lies in the plane. Since $\vec{N}$ is $\perp$ to the plane, $\vec{N} \perp \overrightarrow{P_0Q}$.
This happens iff $\vec{N} \cdot \overrightarrow{P_0Q} = 0$

$$\Rightarrow \quad \vec{N} \cdot (\overrightarrow{OQ} - \overrightarrow{OP_0}) = 0$$
$$\Rightarrow \quad (\vec{N} \cdot \overrightarrow{OQ}) - (\vec{N} \cdot \overrightarrow{OP_0}) = 0$$
$$\Rightarrow \quad \vec{N} \cdot \overrightarrow{OQ} = \vec{N} \cdot \overrightarrow{OP_0}.$$
$$\Rightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

If we write it out, we have the plane is the set of points (x,y,z) where

$$\boxed{ax + by + cz = d} \quad \text{where } d = ax_0 + bx_0 + cz_0$$

Example: ① Find an eqn of the plane passing through $(9,-2,5)$ with normal vector $\vec{N} = (-6,-6,1)$.

② Find an equation of the plane containing the lines:
$$\vec{l}_1(t) = (1,1,1) + t(2,0,-1)$$
$$\vec{l}_2(t) = (1,1,1) + t(-1,-1,7).$$

Sol: ① Here we have $P_0 = (9,-2,5)$ and $\vec{N} = (-6,-6,1)$.
So the eqn is.
$$\begin{aligned} -6x - 6y + 1z &= \overrightarrow{OP_0} \cdot \vec{N} \\ &= (9)(-6) + (-2)(-6) + (5)(1) \\ &= -36 + 12 + 5 \\ &= -19 \end{aligned}$$
$$\Rightarrow \quad \boxed{-6x - 6y + z = -19}$$

② We need a pt. P_0 on the plane and a vector $\vec{N}$ that is normal to the plane.

By inspection, we can see $P_0 = (1,1,1)$ is on the plane since it is on the lines.

Earlier, we used the cross product to find that the vector $\vec{N} = (2,0,1) \times (-1,-1,7) = (1,-13,2)$ is $\perp$ to the plane.

So an eqn for the plane is:

$$1x - 13y + 2z = \vec{N} \cdot \overrightarrow{OP_0}$$
$$= (1)(1) + (-13)(1) + (2)(-1)$$
$$= 1 - 13 - 2$$
$$= -14$$

$$\Rightarrow \quad \boxed{x - 13y + 2z = -14}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 23, 2017

Section:

- [ ] §2.1

Topics Covered:

- [ ] Multivariable functions
- [ ] Graphs of functions
- [ ] Level sets

§2.1: Functions / Graphs / Level sets:

Functions:

Right now, you should be very comfortable with functions f, whose domain is some subset, U, of $\mathbb{R}$, and whose range is some subset of $\mathbb{R}$. We denote such fcns by

$$\boxed{\begin{array}{l} f: U \subseteq \mathbb{R} \longrightarrow \mathbb{R}. \\ \qquad\quad x \longmapsto y = f(x) \end{array}}$$

In this class, we consider multivariable functions f whose domain, U, is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a subset of $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$. We will greatly empasize functions from $\mathbb{R}^2$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{R}^2$ or $\mathbb{R}^3$.

<u>Example:</u>
1. $f(x,y) = x^2 + y^2$, $f: \mathbb{R}^2 \to \mathbb{R}$
2. $g(x,y,z) = \sqrt{xyz}\, e^{x+y+z}$, $g: \{(x,y,z) \mid xyz \geq 0\} \subseteq \mathbb{R}^3 \to \mathbb{R}$
3. $\vec{\ell}(t) = (1,2,3) + t(-1,0,5)$, $\vec{\ell}: \mathbb{R} \to \mathbb{R}^3$.

Examples 1 and 2 are called scalar valued because the output is a scalar. Example 3 is vector valued.

Graphs:

In order to understand the behavior of a function, we usually try to visualize the function in the form of a graph.

Def: The graph of a function $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the set of points of the form $(x, y, f(x,y))$ where (x,y) is a point in U.

Note: ① The graph of a fcn $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$.

② We can define graphs of functions $g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where n, m are any positive integers similarly.

As we will see, graphs of multivariable fcns are much harder to visualize and draw than single variable fcns. We now discuss tools to help us graph.

Example: $f(x,y) = x^2+y^2$.

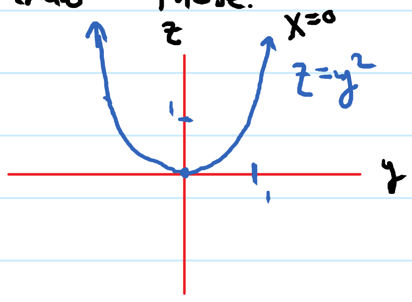
Idea: Instead of trying to graph immediately, we can look at 2-dim. cross sections. We use these to create a wire frame, which will be the skeleton for our graph.

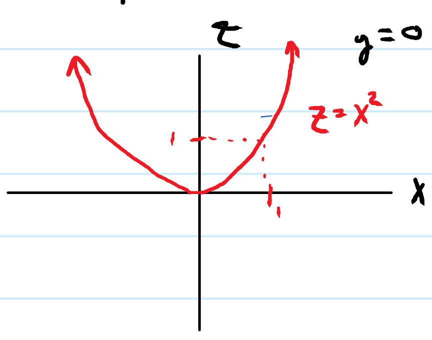
To make cross sections, we fix one of the variables.

Fix $x=0$, then $z = f(0,y) = 0^2 + y^2 = y^2$ is the intersection of the graph with the yz-plane.

Fix $y=0$, then $z = f(x,0) = x^2 + 0^2 = x^2$ is the intersection of the graph with the xz-plane.

These are both just parabolas! We know how to draw those.





Let's try fixing $z=c$ for varying constants c.

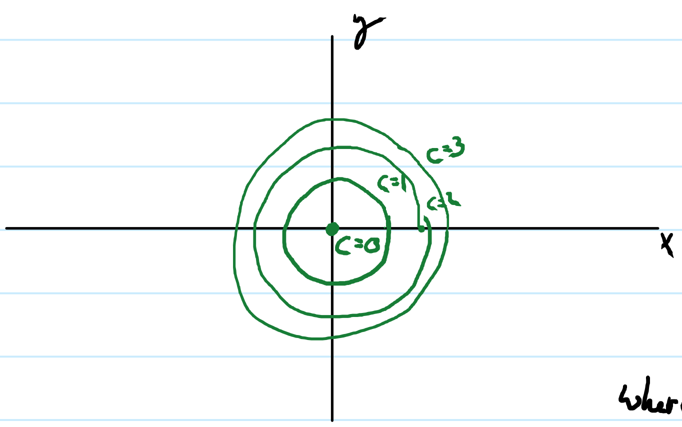
If $c=0$, $0 = f(x,y)$
$0 = x^2+y^2$
the set of (x,y) where $f(x,y)=0$ is $\{(0,0)\}$.

If $c=1$, $1 = f(x,y)$
$1 = x^2+y^2$
This is the unit circle!

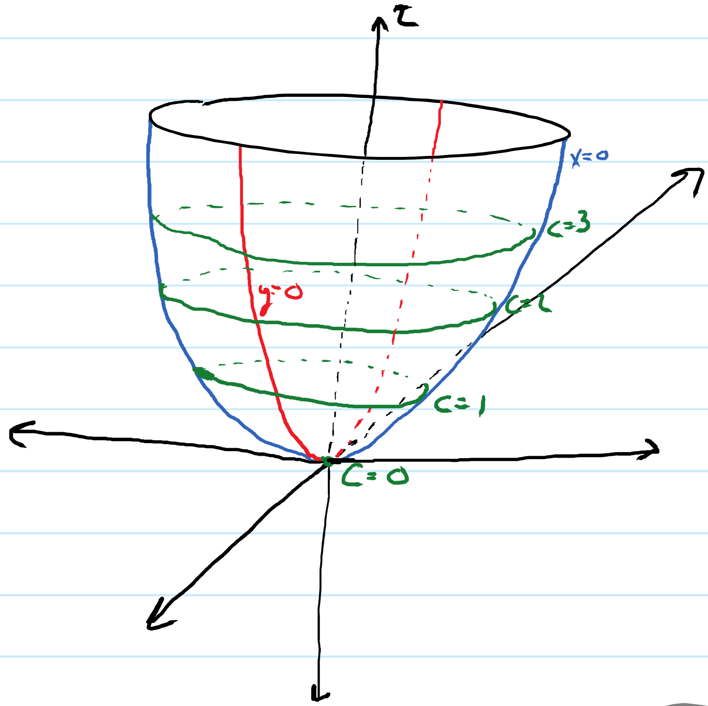
If $c=2$, $2 = f(x,y)$
$2 = x^2+y^2$
Circle based at the origin with radius $\sqrt{2}$.

If $c=-1$, $-1 = f(x,y)$
$-1 = x^2+y^2$
This is impossible since $x^2+y^2 \geq 0$.

Now, we draw these curves in the xy-plane.



To make the wire frame, imagine the z-axis coming out of the board. Since the circle where $c=1$ is the set of points on the graph where $z=1$, imagine you "pull" the $c=1$ circle out of the board by 1 unit. Pull the circle where $c=2$ out of the board by 2 units, and so on. Before we even sketch it, maybe you can visualize a "bowl shape". This graph is called a paraboloid.



paraboloid.

Notice the graph lies entirely above the xy-plane because $x^2+y^2=C$ is impossible if $C<0$.

Def: The set of points, (x,y), where $f(x,y)=c$ is called the level set or level curve of heigt c (or value c).

Example: Draw level curves of height $-2, -1, 0, 1, 2$ for $f(x,y)=x^2-y^2$. Use them to graph f.

Sol: $c=0$:

$$0=f(x,y)$$
$$0=x^2-y^2$$
$$y^2=x^2$$
$$y=\pm x \quad \longleftrightarrow \text{ intersection of two lines through the origin.}$$

$c=1$

$$1=f(x,y)$$
$$1=x^2-y^2 \quad \longleftrightarrow \text{ hyperbola "opening horizontally"}$$

$c=2$

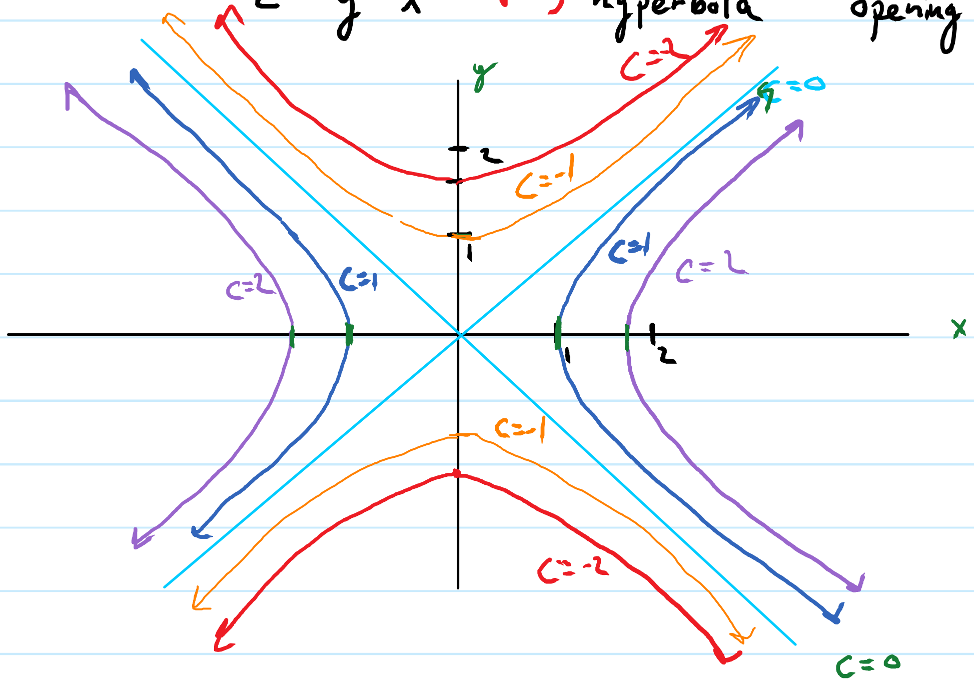
$$2=x^2-y^2 \quad \longleftrightarrow \text{ hyperbola "opening horizontally"}$$

$c=-1$

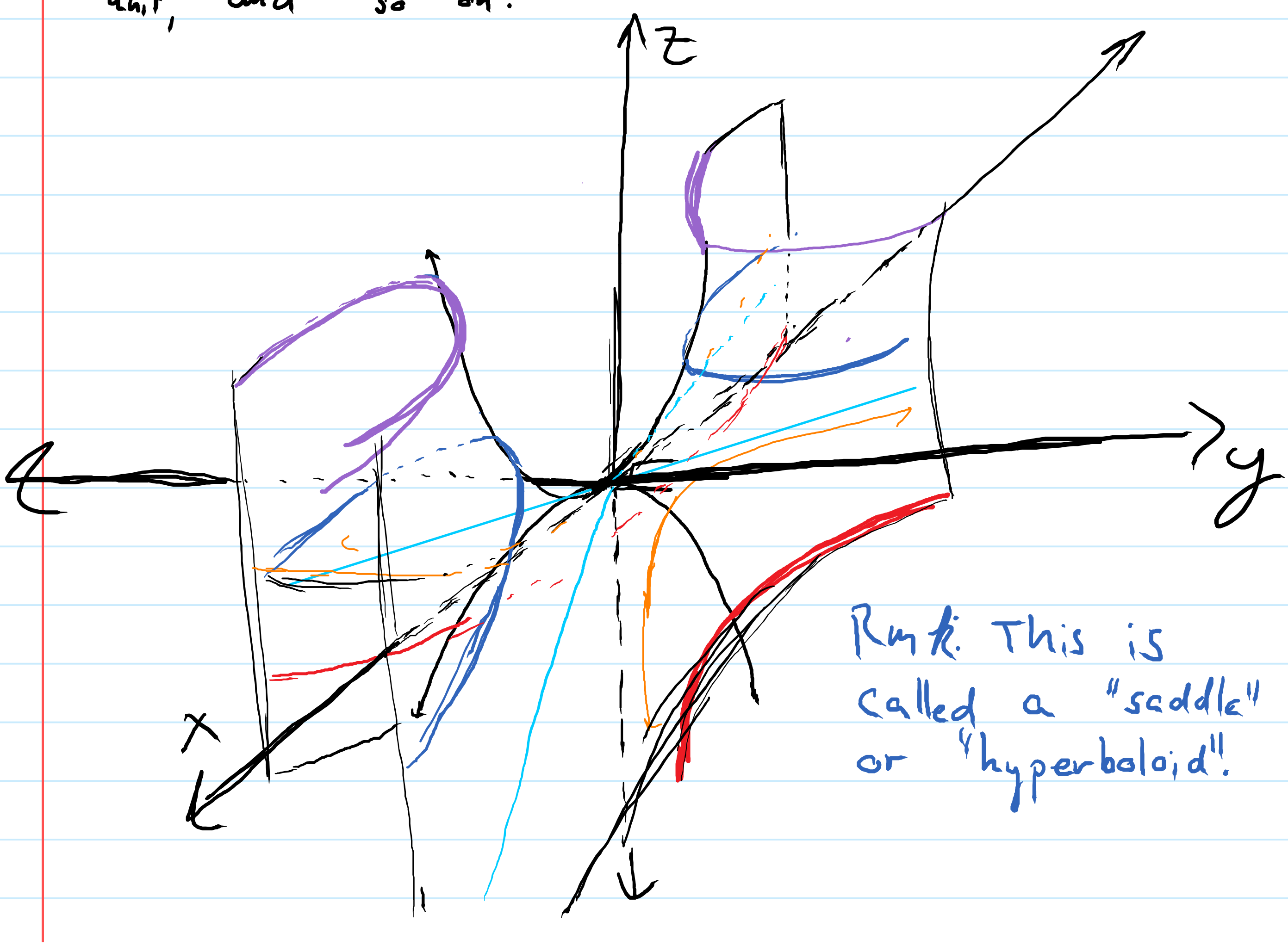
$$-1=f(x,y)$$
$$-1=x^2-y^2$$
$$1=y^2-x^2 \quad \longleftrightarrow \text{ hyperbola "opening vertically"}$$

$c=-2$: $\quad -2 = x^2 - y^2$

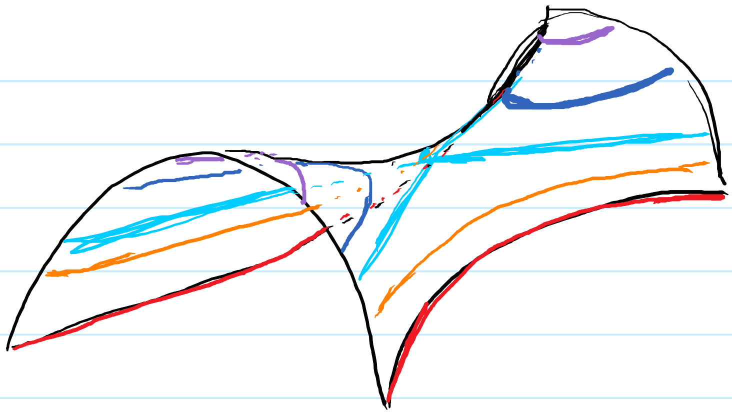
$2 = y^2 - x^2 \longleftrightarrow$ hyperbola "opening vertically"



Then we pull the curve $c=1$ out of the board 1 unit, push the curve $c=-1$ into the board 1 unit, and so on.



Rmk. This is called a "saddle" or "hyperboloid".



More rough drawing.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 25, 2017

Section:

- [ ] §2.2

Topics Covered:

- [ ] Limits and continuity

§2.2: Limits and Continuity:

Before discussing limits of multivariable functions, let's review limits of single variable functions.

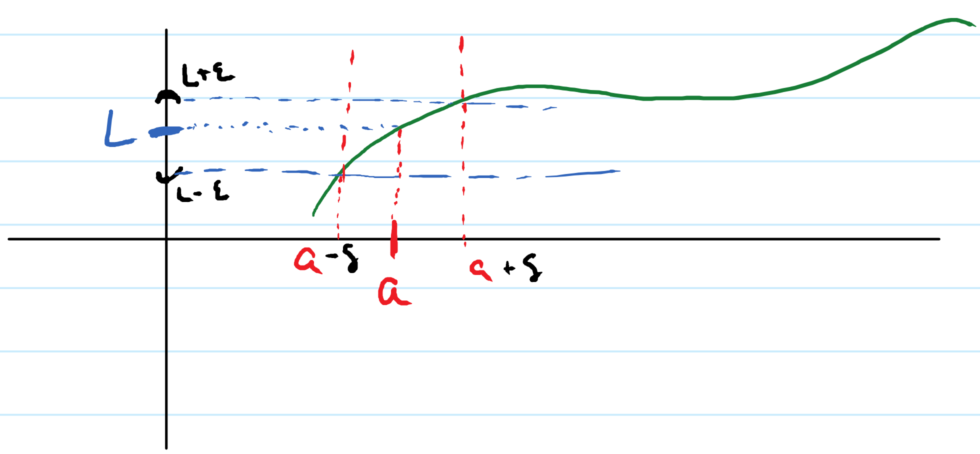
Def: Let f be a fcn of one variable, let a be in the domain of f. We say

$$\lim_{x \to a} f(x) = L \qquad \text{if for every}$$

positive number ε, there is a positive number δ so that

$$L - \varepsilon < f(x) < L + \varepsilon \qquad \text{whenever}$$
$$a - \delta < x < a + \delta.$$

What this means intuitively is that I can force $f(x)$ to be as close as I want to L by picking x values that are very close to a. Equivalently, this means that as x gets closer and closer to a, $f(x)$ gets closer and closer to L.



We use this definition to motivate the definition of limits of multivariable fcns. We want to say $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $f(x,y)$ gets very close to L when (x,y) gets very close to (a,b).

Definition: Let $f: \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$. Let (a,b) be a point in $\mathcal{U}$ (the domain of f). We say

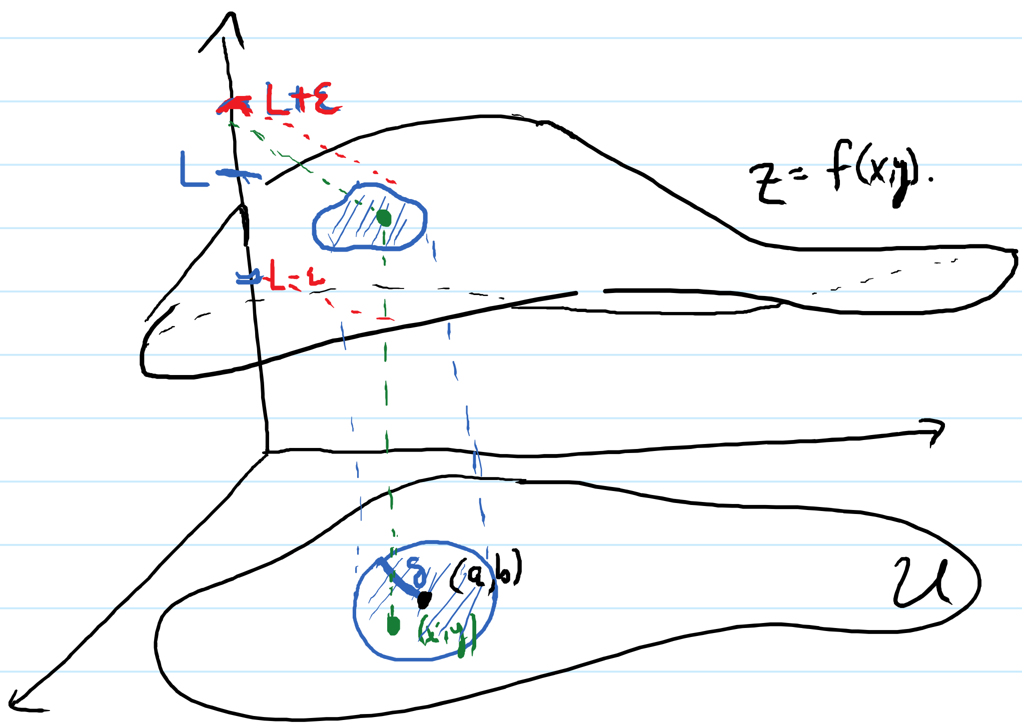
$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for every positive real number ε, there is a positive number δ so that:

$$|f(x,y) - L| < \varepsilon \quad \text{whenever}$$

distance from $f(x,y)$ to L.

distance from (x,y) to (a,b)

$$\|(x-a, y-b)\| < \delta$$

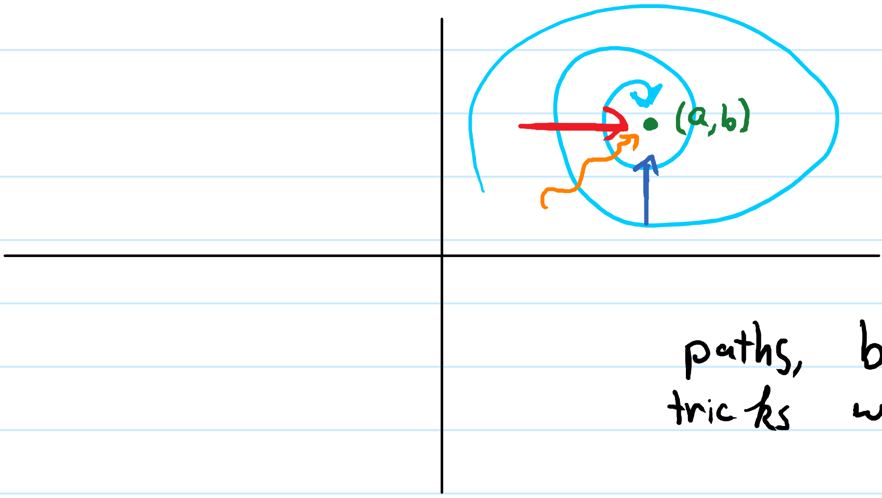


Examples: (1) $\lim_{(x,y)\to(1,2)} \frac{x^2}{x+y} = \frac{1^2}{1+2} = \boxed{\frac{1}{3}}$

(2) $\lim_{(x,y)\to(1,\frac{\pi}{4})} \frac{\cos(y)}{x} = \frac{\cos(\frac{\pi}{4})}{1} = \boxed{\frac{\sqrt{2}}{2}}$

Problem: For fcns $f:\mathbb{R}\to\mathbb{R}$, $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$, and $\lim_{x\to a^-} f(x)$ exist and are equal. I.e., approaching a from the left or right yields the same result.

If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$, and if (a,b) is a point in U, in order for $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, we must get the same limit no matter how we approach the point (a,b). Since (a,b) is in $\mathbb{R}^2$, there could be infinitely many different ways to approach (a,b)! As a result, it is much harder to compute $\lim_{(x,y)\to(a,b)} f(x,y)$, or to even show it exists,



(different paths to (a,b)).

We cannot possibly check all paths, but there are some tricks we can use.

Proving a limit does not exist:

To show a limit does not exist, approach the point along different paths and get different answers.

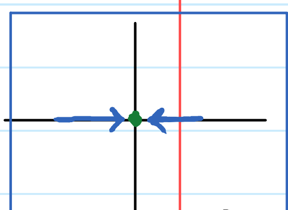
Example: ① $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x(y+1)}{x+y}$.

$\boxed{\text{Notice that we can't just plug in } (0,0)!}$

If we approach $(0,0)$ along the path where $x=0$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x(y+1)}{x+y} = \lim_{y\to 0} \frac{0\cdot(y+1)}{0+y} = \lim_{y\to 0} \frac{0}{y} = \lim_{y\to 0} 0 = 0.$$

If we approach $(0,0)$ along the path where $y=0$, we get



$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x(y+1)}{x+y} = \lim_{x\to 0} \frac{x(0+1)}{x+0} = \lim_{x\to 0} \frac{x}{x} = 1.$$

Since these don't match, the limit does not exist.

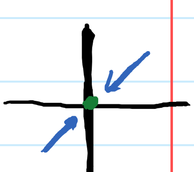
---

② $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{2x^2+3y^2}$.

If we approach $(0,0)$ along the path where $x=0$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{3xy}{2x^2+3y^2} = \lim_{y\to 0} \frac{3(0)\cdot y}{2(0^2)+3y^2} = \lim_{y\to 0} \frac{0}{3y^2} = \lim_{y\to 0} 0 = 0.$$

If we approach $(0,0)$ along the path where $y=0$, we again get 0. However, if we approach $(0,0)$ along the path where $x=y$, we get:

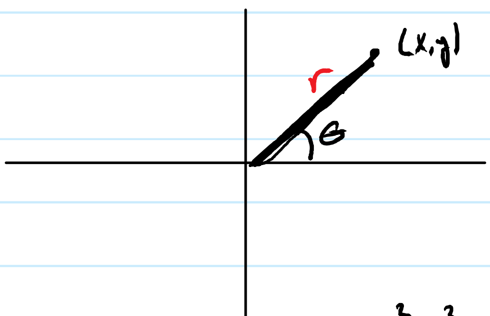


$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{3xy}{2x^2+3y^2} = \lim_{x\to 0} \frac{3x\cdot x}{2x^2+3x^2} = \lim_{x\to 0} \frac{3x^2}{5x^2} = \frac{3}{5}.$$

Since we didn't get the same thing every time, the limit does not exist.

The main techniques for computing limits: squeeze Thm & polar:

Using polar coordinates can be helpful if you are computing limits at the origin. Recall: If (x,y) is a point in $\mathbb{R}^2$, $x = r\cos\theta$ $y = r\sin\theta$ where r is the distance from (x,y) to $(0,0)$, and θ is the angle that the vector (x,y) makes with the positive x-axis.



Since $(0,0)$ is the only point where $r=0$, "$(x,y)\to(0,0)$" simplifies to "$r\to 0$".

Examples: ① $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos^2\theta + r^2\sin^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$

$$= \lim_{r\to 0} \frac{r^2(\cos^2\theta + \sin^2\theta)}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \lim_{r\to 0} \frac{r^2}{\sqrt{r^2}}$$

$$= \lim_{r\to 0} \frac{r^2}{r}$$

$$= \lim_{r\to 0} r$$

Since this final expression does not depend on θ,

the limit is 0 as $(x,y)\to(0,0)$ along any path. Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$

(2) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r}$$

$$= \lim_{r\to 0} r\cos\theta\sin\theta$$

Now since $-1 \leq \cos\theta, \sin\theta \leq 1$ for all θ

$$-r \leq r\cos(\theta)\sin(\theta) \leq r$$

$$\lim_{r\to 0}(-r) \leq \lim_{r\to 0} r\cos\theta\sin\theta < \lim_{r\to 0} r$$

$$\Rightarrow \quad 0 \leq \lim_{r\to 0} r\cos(\theta)\sin(\theta) \leq 0$$

$$\Rightarrow \lim_{r\to 0} r\cos\theta\sin(\theta) = 0$$

The technique in the blue box is called the Squeeze Thm.

Example: (Squeeze Thm example).

$$\lim_{(x,y)\to(0,0)} xy\tan^{-1}\left(\frac{1}{e^{xy}}\right).$$

Hint: Use Squeeze Thm when you have something "simple" times something "wild" but bounded.

The range of $f(x) = \tan^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Therefore, for any x, y, $-\frac{\pi}{2} \leq \tan^{-1}\left(\frac{1}{e^{xy}}\right) \leq \frac{\pi}{2}$.

So $$\left(\frac{-\pi}{2}\right)xy \leq xy\tan^{-1}\left(\frac{1}{e^{xy}}\right) \leq \frac{\pi}{2}xy$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \left(-\frac{\pi}{2}\right)xy \leq \lim\limits_{(x,y)\to(0,0)} xy\tan^{-1}\left(\frac{1}{e^{xy}}\right) \leq \lim\limits_{(x,y)\to(0,0)} \left(\frac{\pi}{2}\right)xy.$

$$\Rightarrow \qquad 0 \leq \lim_{(x,y)\to(0,0)} xy\tan^{-1}\left(\frac{1}{e^{xy}}\right) \leq 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} xy\tan^{-1}\left(\frac{1}{e^{xy}}\right) = 0.$$

Using limits, we can define continuity.

Def: If $\vec{x}_0$ is a point in the domain of f, f is said to be continuous at $\vec{x}_0$ if $\lim\limits_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$. f is continuous if $\lim\limits_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$ for all x_0 in the domain of f.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 27, 2017

Section:

- [ ] §2.3 (part I)

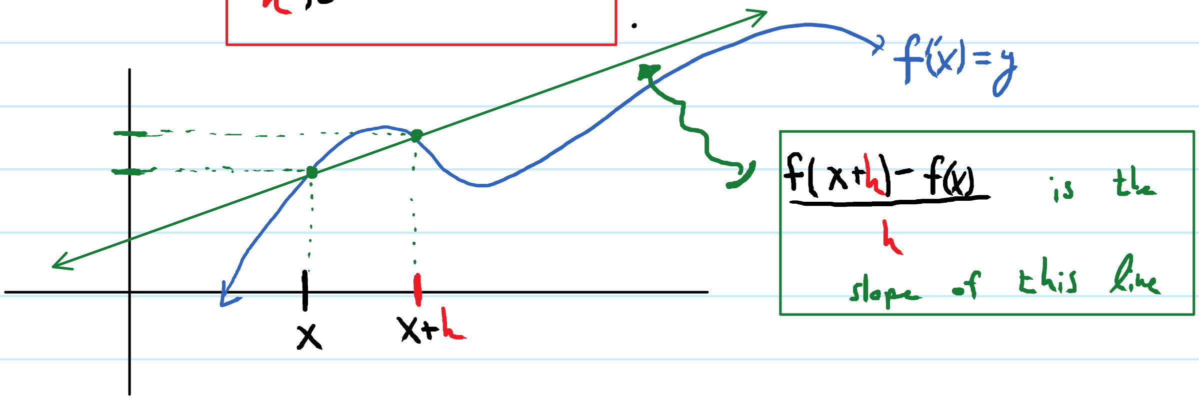
Topics Covered:

- [ ] Partial Derivatives
- [ ] Equation of tangent plane

§2.3 Part 1: Partial derivatives:

Review: If $f: U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a single variable fcn, we define the derivative of f (if it exists) by

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$



$f'(x)$ tells you the instantaneous rate of change of f with respect to a (infinitesimally small) change. I.e., $f'(x)$ is the slope of the tangent line of f at x. To define $f'(x)$, take slope of the secant lines, and let the change in x, h, go to 0.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Problem: In single variable calculus, we measure change in f as the input changes. However, there is only one direction that the input can change (left or right). Now, if we have a fcn $f: \mathbb{R}^2 \to \mathbb{R}$, and a point $\vec{x}_0 = (a, b)$, it is not clear what we mean by "the derivative of f" since there are infinitely many directions that we can deviate from the point $\vec{x}_0$, and each choice of direction could very well give us a different change in f.

A modest starting point for us will be:

Partial Derivatives: For the sake of visualizing the geometry, assume, for now, $f: \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Idea: Since f has to input values: x, y, we will start by measuring the change in f w.r.t. a small change in the x or y- variable, while the other variable remains fixed.

Def: Let $f: \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$. The x partial derivative / y partial derivative of f (or partial of f w.r.t. x / y) is defined by:

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \boxed{\lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}} \quad , \quad \boxed{\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}}$$

More generally, if

$$f: \mathbb{R}^n \to \mathbb{R}$$
$$\vec{x} = (x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n)$$

We can take partial derivatives w.r.t. any variable

$$\boxed{f_{x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}}$$

To actually calculate partial derivatives (w.r.t. say x) just pretend all the other variables are constant and derive as usual.

<u>Examples:</u> (1) $f(x,y) = x^2y - 2xy\ln(y)$.

- $\frac{\partial f}{\partial x} = 2xy - 2y\ln(y)$
- $\frac{\partial f}{\partial y} = x^2 - 2x\ln(y) - 2xy\left(\frac{1}{y}\right)$ ⟵⟶ Product rule!

$= x^2 - 2xy\ln(y) - 2x$

(2) $g(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$.

- $\frac{\partial g}{\partial z} = \frac{(x^2+y^2+z^2)(xy) - (xyz)(2z)}{(x^2+y^2+z^2)^2}$ ⟵⟶ Quotient Rule.

- $\frac{\partial g}{\partial x} = \frac{(x^2+y^2+z^2)(yz) - (xyz)(2x)}{(x^2+y^2+z^2)^2}$

- $\frac{\partial g}{\partial y} = \frac{(x^2+y^2+z^2)(xz) - (xyz)(2y)}{(x^2+y^2+z^2)^2}$

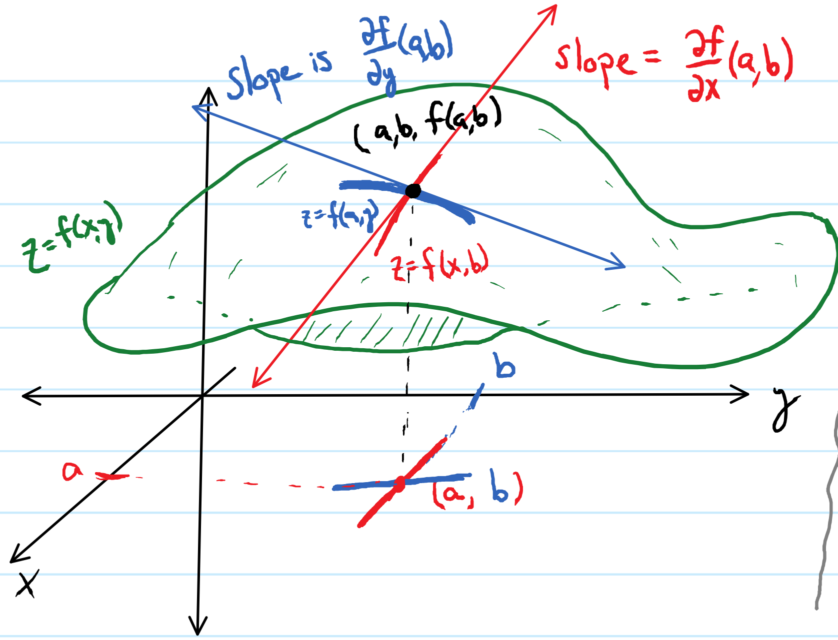
<u>Example:</u> with g as above:

$$\frac{\partial g}{\partial z}(1,1,0) = \frac{(1^2+1^2+0^2)(1\cdot 1) - ((1)(1)(0))(2(0))}{(1^2+1^2+0^2)^2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$\boxed{Q?}$ What do the partials tell us?

Let $f: \mathbb{R}^2 \to \mathbb{R}$, and fix a point $\vec{x}_0 = (x, b)$ in the domain of f. $\frac{\partial f}{\partial x}(a,b)$ tells us the (instantaneous) rate of change of f as x changes and $y=b$ remains fixed.



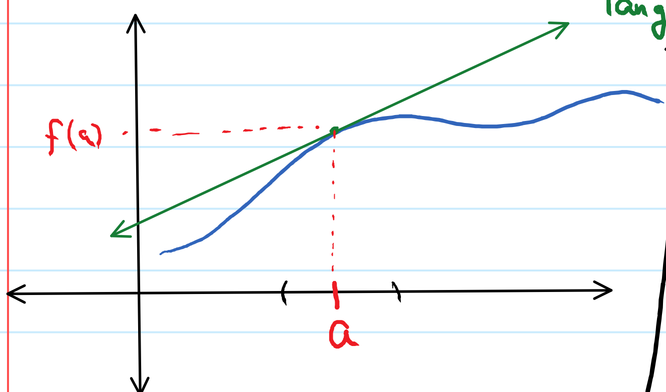
If we look at the points near (a,b) where x varies and $y=b$ is fixed, we have a line segment in the xy-plane $\parallel$ to x-axis. If we plug those points into f, we get a 2D curve sitting on the graph of f. The slope of the tangent line to that curve at (a,b) is the slope of that curve. Similarly, the slope of the blue line is $\frac{\partial f}{\partial y}(a,b)$.

Since the graph of $f(x,y)$ is a 2D-shape sitting in $\mathbb{R}^3$, these two (1D) lines don't tell the whole story of how f changes as you deviate from (a,b). So $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ individually "aren't enough".

$\boxed{Q?}$ What should we call the "derivative of f"? What does it mean for a fcn $f:\mathbb{R}^2\to\mathbb{R}$ to be differentiable?

Let's look back at the single variable case:

Tangent Line: $L(x) = f(a) + f'(a)(x-a)$.



A fcn $f:\mathbb{R}\to\mathbb{R}$ is differentiable at a if the following limit exists

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

What does this tell us?

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \Rightarrow \lim_{x\to a} f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a)$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - \frac{f'(a)(x-a)}{x-a}$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x) - (\underbrace{f(a) + f'(a)(x-a)}_{L(x)})}{x-a}$$

$$\Rightarrow \boxed{0 = \lim_{x\to a} \frac{f(x) - L(x)}{x-a}}$$

In plain English, this says as x gets close to a, $f(x)$ gets very close to $L(x)$, the fcn whose graph is the tangent line of f at a. I.e., the tangent line at a is a good approximation for f.

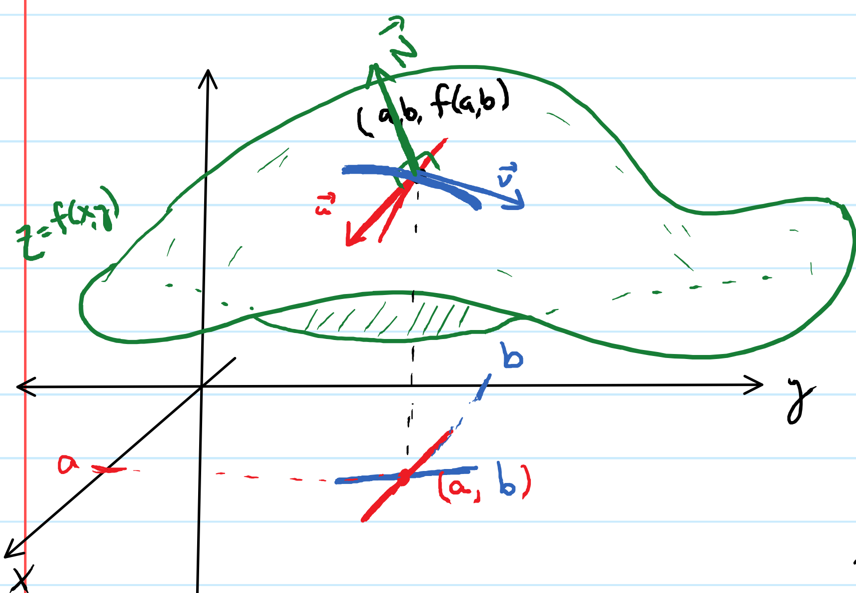
Using this as motivation, we would like to say $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if the tangent plane of f at (a,b) is a good approximation for $f(x,y)$. Warning: It is possible for a fcn to have all partial derivatives, and still not be differentiable. Existence of partial derivatives is <u>not</u> <u>enough</u>.

Computing the equation of the Tangent plane of $f(x,y)$ at (a,b).

As we have seen, in order to get the eqn of the tangent plane, we need: ① A point on the plane: P_0
② A vector $\perp$ to the plane: $\vec{N}$

Since the tangent plane touches the graph of f at $(a, b, f(a,b))$. We let $P_0 = (a, b, f(a,b))$.

To find $\vec{N}$, we will find two vectors in the plane, and cross them (just like before). We need two vectors tangent to the graph.



Let

$\vec{u} = (1, 0, \frac{\partial f}{\partial x}(a,b))$

$\vec{v} = (0, 1, \frac{\partial f}{\partial y}(a,b))$.

These vectors work because $\frac{\partial f}{\partial x}$ is the slope of the tan. line of the red curve on the plane, so $\vec{u}$ is the direction vector of that tangent line. Similarly, $\vec{v}$ is the direction vector of the line tangent to the blue curve $f(a,y)$. So we take

$\vec{N} = \vec{u} \times \vec{v}$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & \frac{\partial f}{\partial x}(a,b) \\ 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & \frac{\partial f}{\partial x}(a,b) \\ 0 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(a,b)\,\vec{i} - \frac{\partial f}{\partial y}(a,b)\,\vec{j} + \vec{k}$$

$$= \left(-\frac{\partial f}{\partial x}(a,b),\ \frac{\partial f}{\partial y}(a,b),\ 1\right)$$

So the plane satisfies the eqn:

$$-\frac{\partial f}{\partial x}(a,b)\,x - \frac{\partial f}{\partial y}(a,b)\,y + z = \vec{N}\cdot\vec{OP_0}$$

$$= -\frac{\partial f}{\partial x}(a,b)\cdot a - \frac{\partial f}{\partial y}(a,b)\cdot b + f(a,b)$$

$$\Rightarrow \boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).}$$

(To be continued...)

Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 26, 2017

Section:
- §2.3 (part 2)

Topics Covered:
- Linear approximation
- The meaning of differentiability, and a criterion for differentiability
- The total derivative

From Last time: If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is "nice" (i.e., the graph is smooth) then the equation $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ should describe the plane tangent to the graph at (a,b).

<u>Example:</u> Find the eqn of the plane tangent to the graph of $f(x,y) = x^2 - 2xy + y^2$ at the point where $x=1$, $y=2$

<u>Sol:</u>

- $\frac{\partial f}{\partial x} = 2x - 2y \Rightarrow \frac{\partial f}{\partial x}(1,2) = 2 - 2 \cdot 2 = 2 - 4 = -2$
- $\frac{\partial f}{\partial y} = -2x + 2y \Rightarrow \partial f/\partial y\,(1,2) = -2(1) + 2(2) = -2 + 4 = 2$
- $f(1,2) = 1^2 - 2(1)(2) + (2)^2 = 1 - 4 + 4 = 1$

$$\Rightarrow \boxed{z = 1 - 2(x-1) + 2(y-2)}$$

<u>Differentiability ($f: \mathbb{R}^2 \to \mathbb{R}$ case) & linear approximation:</u>

Let $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$, (a,b) be a point in the domain of f. Assume $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ exist.

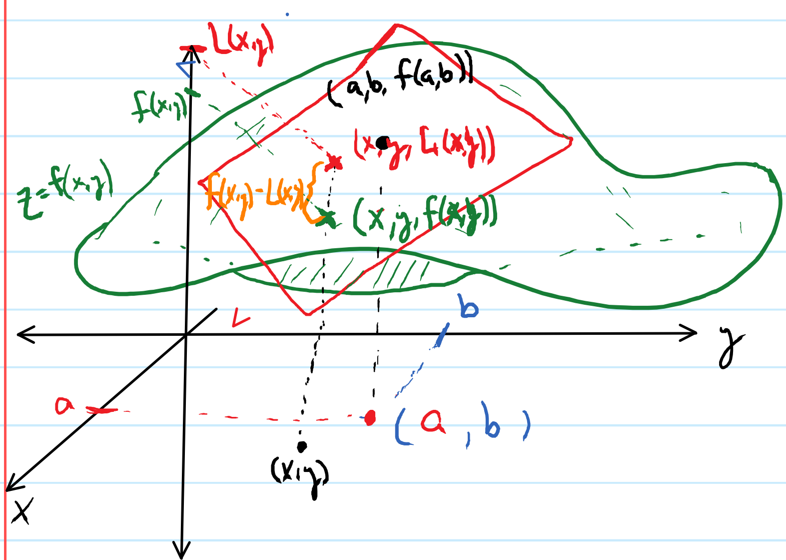
Let $L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$.

$L(x,y)$ is the fcn whose graph is the plane tangent to the graph of f at (a,b).

We saw last time that a fcn of one variable, $g(x)$, differentiable at a point $x=a$ iff the tangent line is a "good approximation" for g. We say $f(x,y)$ is differentiable at (a,b) if $f(x,y) \approx L(x,y)$ if (x,y) is close to (a,b). More precisely:

Definition: In the above setting. f is differentiable at (a,b) if

$$(*) \quad \boxed{\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L_1(x,y)|}{\|(x,y)-(a,b)\|} = 0}$$



Rmk: We will call $L(x,y)$ the linear approximation of f near (a,b)

Problem: It is hard to check property $(*)$ directly.
Luckily, we have:

Thm: If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous "near" (a,b), then f is differentiable at (a,b).

Rmk: A fcn that has continuous partials is called C^1.
This thm says $C^1 \Rightarrow$ differentiable.

Example: Estimate $(0.99)^2 - 2(0.99)(2.01) + (2.01)^2$ using linear approximation.

Sol: Let $f(x,y) = x^2 - 2xy + y^2$. We want to estimate $f(0.99, 2.01)$.
To do this we use Linear approximation near $(1,2)$.

We saw earlier: $\frac{\partial f}{\partial x} = 2x - 2y$, $\frac{\partial f}{\partial y} = -2x + 2y$, which are continuous everywhere. By the Thm, f is diff'ble, so $f(x,y) \approx L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$

near $(1,2)$. We calculated earlier:

$$L(x,y) = 1 - 2(x-1) + 2(y-2).$$

Since $(0.99, 2.01)$ is close to $(1,2)$

$$f(0.99, 2.01) \approx L(0.99, 2.01) = 1 - 2(-0.01) + 2(0.01) = 1 + 0.02 + 0.02 = \boxed{1.04}$$

Rmk: Existance of partials is not good enough for differentiability.
for ex. $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ has partials everywhen,

but is not diff'ble at $(0,0)$ (see book pg. 114 for picture).

Differentiability (general case) and the total derivative (a survey):

Let $f(x,y)$ be diff'ble at (a,b), $L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ be the linear approximation near (a,b).

Let $\nabla f(a,b) = \begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \end{bmatrix}$ be the 1×2 matrix

of partial derivatives. Then, (*) $L(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b)$

dot product

This should look familiar: If $g(x)$ is a one variable diff'ble fcn, $y = g(a) + g'(a)(x-a)$ describes the tangent line at $x=a$.
In (*), $\nabla f(a,b)$ is playing the same roll as the derivative in the single variable case!

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called the gradient of f.

Rmk: Intuitively, in order to fully encapsulate how f changes w.r.t. to a deviation from (a,b), we need to know how f changes w.r.t. x and y. So we need a matrix to hold multiple pieces of information.

More generally: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Then if $\vec{x} = (x_1, x_2, \dots, x_n)$ is an arbitrary point in the domain, we write

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}))$$

each of these is a function of n-variables

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = (\underbrace{2x+y}_{f_1}, \underbrace{3xy}_{f_2}, \underbrace{x^2-y^2}_{f_3})$

Assume each of the f_i's has continuous partial derivatives for each input variable. Then the theorem discussed before says f is differentiable. What does this mean?

Let $\vec{a} = (a_1, \dots, a_n)$ be a vector in $\mathbb{R}^n$ (the domain of f), and let

$$[Df](\vec{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}) & \cdots & & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$$

Then for all points $\vec{x}$ "close to" $\vec{a}$ (i.e., $\|\vec{x}-\vec{a}\|$ is small),

$$(**)\qquad f(x) \approx f(\vec{a}) + [Df](\vec{a})\cdot(\vec{x}-\vec{a})$$

Matrix multiplication

In this equation, we view these as column vectors so that matrix multiplication makes sense.

Here, the matrix of partial derivatives $[Df](\vec{a})$ is called the (total) derivative or the differential of f at $\vec{a}$.

In the special case $m=1$, and we have $f:\mathbb{R}^n \to \mathbb{R}$,

$[Df](\vec{x})$ is just the gradient: ∇f.

Rmk: ① We will use the total derivative when we talk about the multivariable chain rule.
② We will only really consider fcns $f:\mathbb{R}^n \to \mathbb{R}^m$ where $n, m \leq 3$.

Example from before:

Let $f:\mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (2x+y,\ 3xy,\ x^2-y^2)$.

Then $$[Df](x,y) = \begin{pmatrix} 2 & 1 \\ 3y & 3x \\ 2x & -2y \end{pmatrix}$$

(**) More precisely:

$$\lim_{\vec{x} \to \vec{a}} \frac{\| f(\vec{x}) - (f(\vec{a}) + [Df](\vec{a}) \cdot (\vec{x} - \vec{a})) \|}{\| \vec{x} - \vec{a} \|} = 0$$

Note that: 1 the numerator is a vector in $\mathbb{R}^m$, so

"$f(\vec{x}) \approx f(\vec{a}) + [Df](\vec{a}) \cdot (\vec{x} - \vec{a})$" means the difference vector has small magnitude.

2 This limit criterion is the official condition for a funtion to be differentiable.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 1, 2017

Section:
- [ ] §2.4

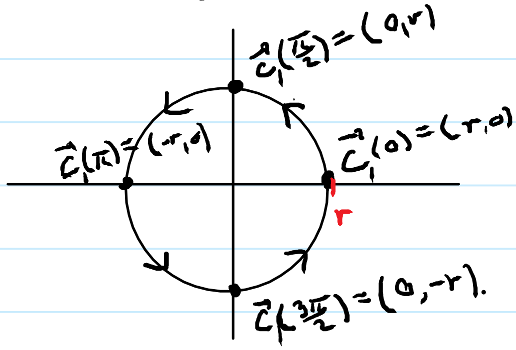
Topics Covered:
- [ ] Paths and curves

§2.4: Paths and Curves:

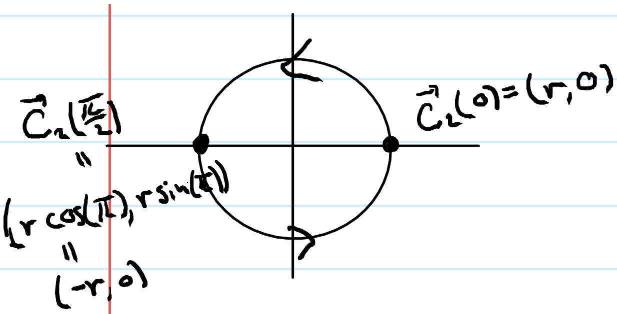
Today we consider the geometry of fcns $\vec{c}:\mathbb{R}\to\mathbb{R}^2$ (or $\mathbb{R}^3$).

For these fcns, we think of the input variable, t, as time. Then for any time value, t_0, $\vec{c}(t_0)$ tells us the position of a particle floating in space. As time progresses, the particle leaves a "dust trail" in its wake. This dust trail will be some curve, C, in 2-space (or 3-space). We say, $\vec{c}$ parametrizes the curve C.

Examples: ① The circle of radius r is given by the equation: $x^2+y^2=r^2$. It can be parametrized by the the fcn
$$\vec{c}_1:\mathbb{R}\to\mathbb{R}^2,\quad \vec{c}_1(t)=(r\cos(t), r\sin(t)).$$
The point $\vec{c}_1(0)=(r\cos(0), r\sin(0))=(r,0)$ is the intersection of the curve with the positive x-axis. As t increases, $\vec{c}(t)$ travels counterclockwise around the circle.



② Let $\vec{c}_2:\mathbb{R}\to\mathbb{R}^2$, $\vec{c}_2(t)=(r\cos(2t), r\sin(2t))$.
$\vec{c}_2(t)$ also traces the circle, but it travels around the circle twice as fast:



This example shows that two different functions can parametrize the same curve.

③ $\vec{c}_3 : \mathbb{R} \to \mathbb{R}^3$, $\vec{c}_3(t) = (\cos(t), \sin(t), t)$.

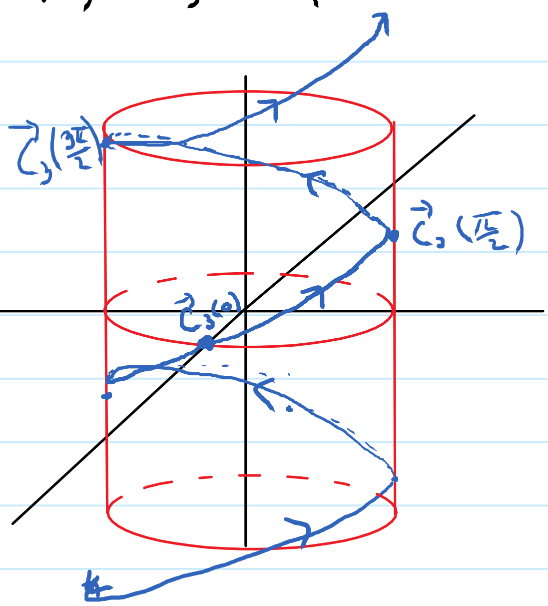
The x and y coordinates always satisfy the equation

$$x^2 + y^2 = 1.$$

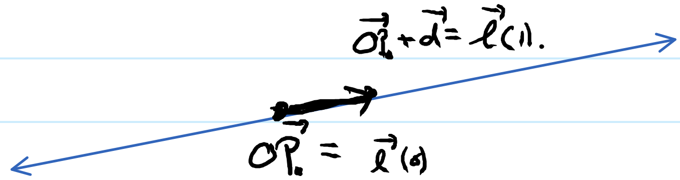
So $\vec{c}_3(t)$ always lies on the cylinder $x^2+y^2=1$

(Note: $x^2+y^2=1$ defines a cylinder in 3-space because the z-coordinate can be any number)

Meanwhile, the z-coordinate rises as time moves on. Therefore, $\vec{c}_3(t)$ parametrizes a helix.



④ If P_0 is a point in $\mathbb{R}^3$, $\vec{d}$ is a direction vector, we've already seen the parametrization of the line $\vec{\ell}(t) = \vec{OP_0} + t\vec{d}$



Velocity vectors:

Since we think of $\vec{c}(t)$ as the position of some particle floating in space, it is reasonable to ask about the speed and direction in which the particle is traveling. We do this with the velocity vector.

Def: If $\vec{c}(t)$ is a path, then the velocity of $\vec{c}$ at time t is defined by

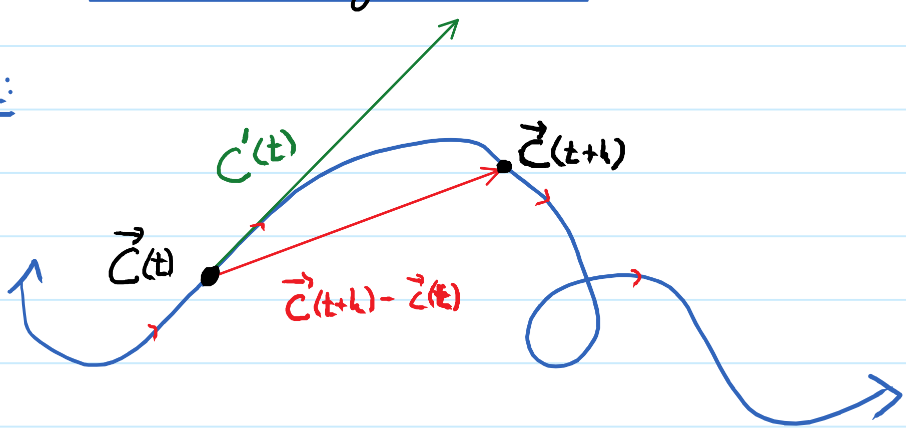
$$\boxed{v(t) = \vec{c}\,'(t) = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}}$$

The speed of $\vec{c}$, denoted $s(t)$ is defined to be

$$\boxed{s(t) = \|\vec{c}\,'(t)\|}$$

It is not hard to see that: ① if $\vec{c}(t) = (x(t), y(t))$, then $\boxed{\vec{c}\,'(t) = (x'(t), y'(t))}$. ② If $\vec{c}(t) = (x(t), y(t), z(t))$, then $\boxed{\vec{c}\,'(t) = (x'(t), y'(t), z'(t))}$.

Picture:



Rmk: ① If we write $\vec{c}: \mathbb{R} \to \mathbb{R}^3$. Then (from §2.3) we have

$$[D\vec{c}](t) = \begin{bmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \\ \frac{dz}{dt}(t) \end{bmatrix}$$

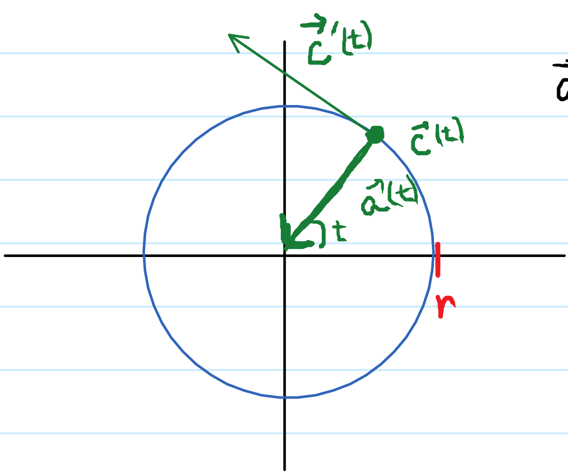
Which is just $\vec{c}'(t)$ but written as a column vector. So this notion of derivative is consistent with the other definition. We like to write $\vec{c}'(t)$ as a row vector because we want to view $\vec{c}'(t)$ as a vector in space.

② It is ok to use the "prime" notation here because there is only one input variable, so there is no ambiguity.

<u>Examples:</u> ① $\vec{c}_1(t) = (r\cos(t), r\sin(t))$ for some constant r.

$$\vec{c}_1'(t) = (-r\sin(t), r\cos(t)).$$

$$\begin{aligned} s(t) = \|\vec{c}_1'(t)\| &= \left((-r\sin(t))^2 + (r\cos(t))^2 \right)^{1/2} \\ &= \left(r^2\sin^2(t) + r^2\cos^2(t) \right)^{1/2} \\ &= (r^2)^{1/2} \\ &= r \qquad \text{(constant speed).} \end{aligned}$$



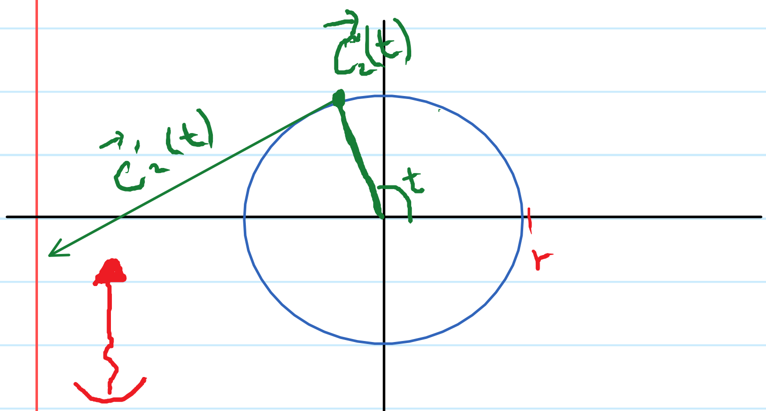
$$\vec{a}(t) = \vec{c}_1''(t) = (-r\cos(t), -r\sin(t))$$

(2) $\vec{C}_2(t) = (r\cos(2t), r(\sin(2t))$

$$c_2'(t) = (-2r\sin(2t),\ 2r\cos(2t))$$

$$s_2(t) = \|c_2'(t)\| = \left((-2r\sin(2t))^2 + (2r\cos(2t))^2 \right)^{1/2}$$
$$= \left(4r^2\sin^2(2t) + 4r^2\cos^2(2t) \right)^{1/2}$$
$$= (4r^2)^{1/2}$$
$$= 2r$$

(constant speed but twice as fast!)

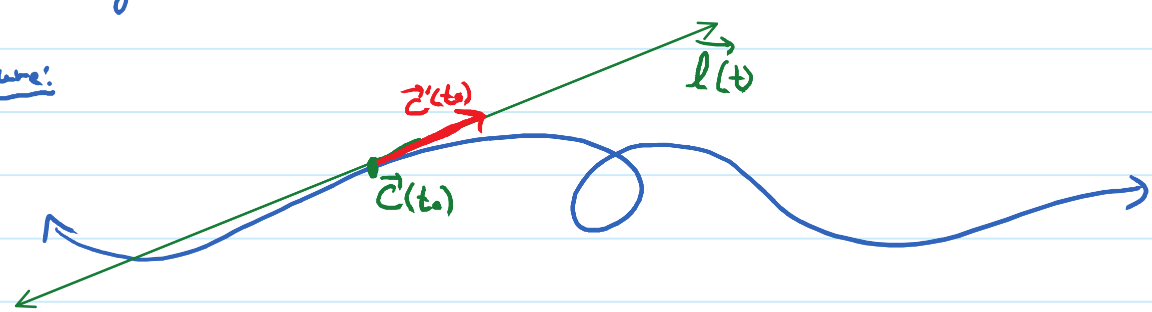


velocity vector is twice as long as $\vec{c}_1'(t)$

Tangent Lines to Paths:

Given a path $\vec{c}(t)$, we can ask for the equation of the line tangent to the C (the curve traced out by $\vec{c}(t)$), at a specific time $t = t_0$.

Picture:



Geometrically, if we imagine the particle "detaching" from the curve C at time t_0, it will fly off and continue along the line $\vec{\ell}(t)$.

Luckily, we already have a point on the line: $\vec{c}(t_0)$, and a direction vector: $\vec{c}\,'(t_0)$. So we could say

$$\vec{\ell}(t) = \vec{c}(t_0) + t\,\vec{c}\,'(t_0).$$

However, we prefer to "shift" the formula so that $\vec{\ell}(t_0) = \vec{c}(t_0)$ (so that we imagine the particle flying off the track at time $t = t_0$).

So the equation we will use is

$$\boxed{\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\,\vec{c}\,'(t_0)}$$

Note: Compare this to the linear approximation equation! It's the same thing again.

Example: Find the equation of the line that is tangent to the path $\vec{c}(t) = (t\cos(t), t\sin(t), t)$ at $t = \pi$.

Sol: Calculate: ① $\vec{c}(\pi) = (\pi\cos(\pi), \pi\sin(\pi), \pi)$
$= (-\pi, 0, \pi)$

② $\vec{c}\,'(t) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t), 1)$
$\vec{c}\,'(\pi) = (\cos(\pi) - \pi\sin(\pi), \sin(\pi) + \pi\cos(\pi), 1)$
$= (-1, -\pi, 1)$. So

$$\boxed{\vec{\ell}(t) = (-\pi, 0, \pi) + (t - t_0)(-1, -\pi, 1).}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 6, 2017

Section:

- [ ] § 2.4 (cont.)
- [ ] § 2.5 (part 1)

Topics Covered:

- [ ] Equation of tangent lines to paths
- [ ] First Properties of the derivative

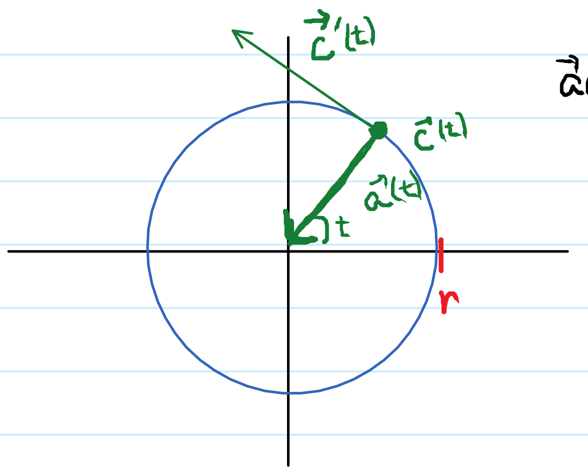
From last Time: If $\vec{c} : \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ is a path. The velocity vector at time t is given by $\vec{v}(t) = \vec{c}\,'(t) = (x'(t), y'(t))$ (or $(x'(t), y'(t), z'(t))$), and the speed of $\vec{c}$ at time t is given by
$s(t) = \|\vec{c}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$ (or $\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$).

We can go one step further, and define the acceleration vector at time t to be $\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$. The acceleration vector tells you in which direction the velocity is changing.

Examples: ① $\vec{c}_1(t) = (r\cos(t), r\sin(t))$ for some constant r.

$$\vec{c}_1\,'(t) = (-r\sin(t), r\cos(t)).$$

$$\begin{aligned} s(t) = \|\vec{c}_1\,'(t)\| &= \left((-r\sin(t))^2 + (r\cos(t))^2 \right)^{1/2} \\ &= \left(r^2\sin^2(t) + r^2\cos^2(t) \right)^{1/2} \\ &= (r^2)^{1/2} \\ &= r \quad \text{(constant speed).} \end{aligned}$$



$$\vec{a}(t) = \vec{c}_1\,''(t) = (-r\cos(t), -r\sin(t))$$

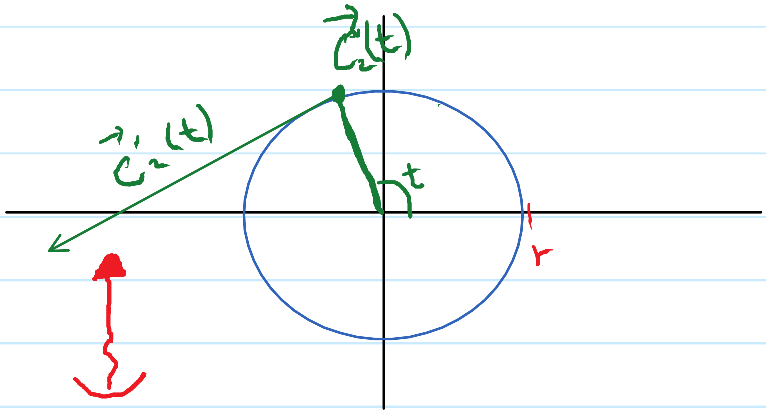
② $\vec{c}_2(t) = (r\cos(2t), r(\sin(2t))$

$$c_2'(t) = (-2r\sin(2t), 2r\cos(2t))$$

$$\begin{aligned} s_2(t) = \|c_2'(t)\| &= \left((-2r\sin(2t))^2 + (2r\cos(2t))^2 \right)^{1/2} \\ &= \left(4r^2\sin^2(2t) + 4r^2\cos^2(2t) \right)^{1/2} \end{aligned}$$

$$= (4r^2)^{1/2}$$
$$= 2r$$

(constant speed but twice as fast!)

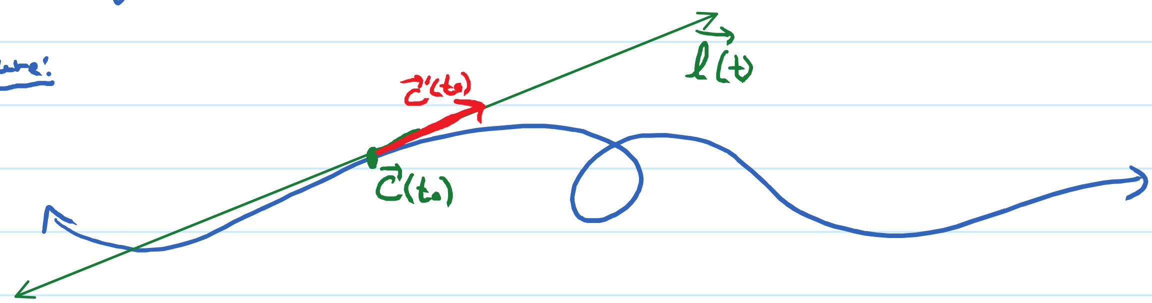


velocity vector is twice
as long as $\vec{c}_1'(t)$

Tangent Lines to Paths:

Given a path $\vec{c}(t)$, we can ask for the equation of the line tangent to the C (the curve traced out by $\vec{c}(t)$), at a specific time $t=t_0$.

Picture:



Geometrically, if we imagine the particle "detaching" from the curve C at time t_0, it will fly off and continue along the line $\vec{\ell}(t)$.

Luckily, we already have a point on the line: $\vec{c}(t_0)$, and a direction vector: $\vec{c}'(t_0)$. So we could say

$$\vec{\ell}(t) = \vec{c}(t_0) + t\,\vec{c}'(t_0).$$

However, we prefer to "shift" the formula so that $\vec{\ell}(t_0) = \vec{c}(t_0)$ (so that we imagine the particle flying off the track at time $t = t_0$).

So the equation we will use is

$$\boxed{\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\,\vec{c}\,'(t_0)}$$

Note: Compare this to the linear approximation equation! It's the same thing again.

Example: Find the equation of the line that is tangent to the path $\vec{c}(t) = (t\cos(t), t\sin(t), t)$ at $t = \pi$.

Sol: Calculate: ① $\vec{c}(\pi) = (\pi\cos(\pi), \pi\sin(\pi), \pi)$

$$= (-\pi, 0, \pi)$$

② $\vec{c}\,'(t) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t), 1)$

$$\vec{c}\,'(\pi) = (\cos(\pi) - \pi\sin(\pi), \sin(\pi) + \pi\cos(\pi), 1)$$

$$= (-1, -\pi, 1). \quad \text{so}$$

$$\boxed{\vec{\ell}(t) = (-\pi, 0, \pi) + (t - t_0)(-1, -\pi, 1).}$$

---

§2.5: Properties of the derivative (with an emphasis on the chain rule):

The main objective in this section will be to explain the multivariable versions of the product rule and the chain rule. Since we discussed the total derivative of a multivariable fcn, the chain rule will look very familiar.

Basic Properties of derivatives:

Let's quickly review some basic operations involving matrices:

① It makes sense to multiply a matrix by a scalar:

Example: $2\begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -2 & 8 \end{bmatrix}$

② If A, B are $n \times m$ matrices (A, B are the same dimension) you can add matrices componentwise:

Example: $\begin{bmatrix} 2 & 0 \\ 1 & 6 \\ -4 & 10 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 20 & 11 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 21 & 17 \\ 1 & 4 \end{bmatrix}$

③ If A is an $n \times m$ matrix and B is an $m \times p$ matrix, then we can multiply AB. The entry of AB in the i^{th} row and j^{th} column is the dot product of the i^{th} row of A with the j^{th} row of B. The result is an $n \times p$ matrix.

Example:

$$2\times 3\left\{ \begin{bmatrix} 1 & -1 & 0 \\ 5 & 7 & -9 \end{bmatrix} \right. \underbrace{\begin{bmatrix} 8 & 8 \\ 0 & 3 \\ -2 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 8 & 5 \\ 58 & 31 \end{bmatrix}.$$

Theorem: (Basic Properties)

① If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble at $\vec{x}_0$, and c is a scalar, $[D(cf)](\vec{x}_0) = c\,[D(f)](\vec{x}_0)$.

(We can pull out scalars).

② If $f,g: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are diff'ble at $\vec{x}_0$, then $f+g: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

$$[D(f+g)](\vec{x}_0) = [Df](\vec{x}_0) + [Dg](\vec{x}_0)$$

These are all $n \times m$ matrices.

③ (Product Rules) ⓘ If $f,g: \mathbb{R}^n \to \mathbb{R}$, then $fg: \mathbb{R}^n \to \mathbb{R}$.
If f,g are diff'ble at $\vec{x}_0$

$$\nabla(fg)(\vec{x}_0) = f(\vec{x}_0)\nabla g(\vec{x}_0) + g(\vec{x}_0)\nabla f(\vec{x}_0)$$

Scalar mult. ($\nabla f, \nabla g$ are n-dim. vectors!)

ⓘⓘ If $\vec{c}(t)$, $\vec{r}(t)$ are paths, then

$$\frac{d}{dt}(\vec{c}(t) \cdot \vec{r}(t)) = \vec{c}\,'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}\,'(t)$$
$$\frac{d}{dt}(\vec{c}(t) \times \vec{r}(t)) = (\vec{c}\,'(t) \times \vec{r}(t)) + (\vec{c}(t) \times \vec{r}\,'(t)).$$

Example: Consider the parametrization of the circle $x^2+y^2=r^2$ from before:

$\vec{c}(t) = (r\cos(t), r\sin(t))$. We observed

before that: ① $\vec{c}$ had constant speed r. I.e.
$s(t) = \|\vec{c}\,'(t)\| = r$ for all t.
② The accelleration vector $\vec{a}(t)$ appeared to be orthogonal to velocity $\vec{v}(t)$.

Let's prove this is indeed true:

Since speed is constant, $\|\vec{c}'(t)\| = r$ is constant, and therefore, $\|\vec{c}'(t)\|^2 = \vec{c}'(t) \cdot \vec{c}'(t) = r$ is constant.

Taking derivatives on both sides, we get

$$\frac{d}{dt}\left(\vec{c}'(t) \cdot \vec{c}'(t)\right) = \frac{d}{dt}(r) = 0 \qquad \text{(derivative of a constant is 0.)}$$

$\Rightarrow \quad \vec{c}''(t) \cdot \vec{c}'(t) + \vec{c}'(t) \cdot \vec{c}''(t) = 0$

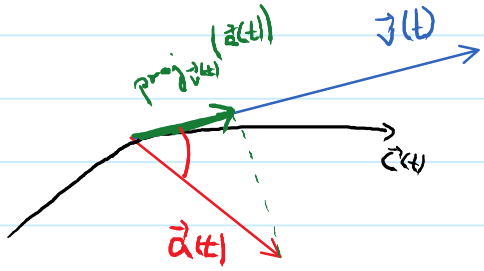
$\Rightarrow \quad 2\left(\vec{c}''(t) \cdot \vec{c}'(t)\right) = 0$

$\Rightarrow \quad \vec{c}''(t) \cdot \vec{c}'(t) = 0$

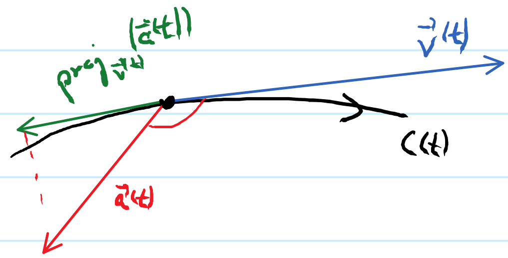
$\Rightarrow \quad \vec{a}(t) \cdot \vec{v}(t) = 0$

$\Rightarrow \quad \vec{a}(t) \perp v(t)$ for all t. ▨

Rmk: Intuitively, this makes sense. If the angle between $\vec{a}(t)$ and $\vec{v}(t)$ were acute, there would be some positive acceleration, which would cause the speed to increase. Similarly if the angle were obtuse, there would be some negative acceleration and the speed would decrease.



accelerating forward
$\Rightarrow$ speed increase.



accelerating backwards
$\Rightarrow$ speed decrease.

Longo: Math 20C - Winter 2017
Lecture Notes

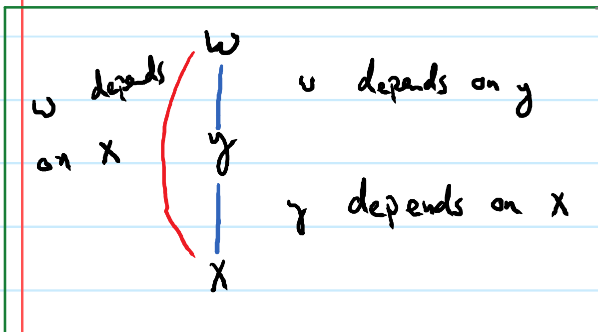
Date: February 8, 2017

Section:
- [ ] §2.5 (part 2)

Topics Covered:
- [ ] The chain rule for multivariable functions

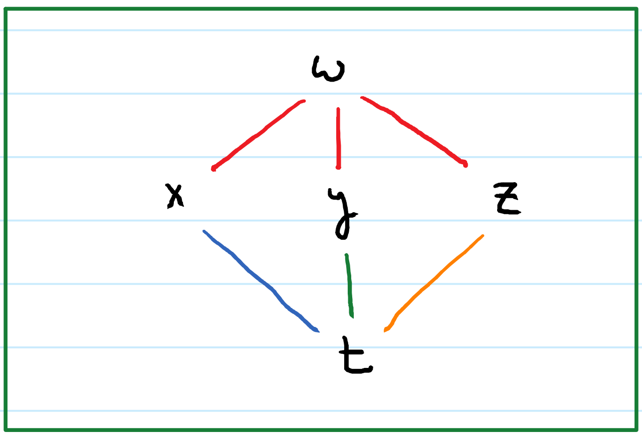
Multivariable Chain Rule:

Let $f, g : \mathbb{R} \to \mathbb{R}$ be diff'ble fcns. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is also diff'ble and $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x).}$ In Leibniz notation, if $y = f(x)$ and $w = g(y)$, we see w is a fcn of x and $\frac{dw}{dx} = \left(\frac{dw}{dy}\right) \cdot \left(\frac{dy}{dx}\right)$.



For multivariable fcns, the situation is more complicated (as expected). Let's start with a simple example.

Suppose w is a function of 3 variables: x, y, z.
Suppose x, y, z all depend on a single variable, t.



Then w is a function of t, so it makes sense to ask how w changes with respect to a change in t. It turns out

$$\boxed{\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial w}{\partial y}\right) \cdot \left(\frac{dy}{dt}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{dx}{dt}\right)}$$

We will discuss where this formula comes from shortly, but for now think of it as the sum of the contributions in change coming from the changes in each of the variables.

Example: A bee flies around the room. Suppose at time $t > 0$, the position of the bee is given by

$\vec{c}(t) = (t^2+1,\ t^3+2t,\ \frac{1}{t})$. Suppose the temperature, T, in Fahrenheit at any point in the room is given by

$$T(x,y,z) = 10e^{-x-2y-4z} + 80.$$

① After 2 seconds, what is the bee's temperature.
② How is the bee's temperature changing at 2 seconds?

$\boxed{\text{Sol:}}$ ① At the 2 second mark, the bee is at the point $\vec{c}(2) = (2^2+1,\ 2^3+2(2),\ \frac{1}{2})$. The bee's temperature is

$$T(\vec{c}(2)) = 10\,e^{-5-24-2} + 80 = \boxed{\frac{10}{e^{-31}} + 80}$$

② The change in the bee's temperature at time $t=2$ is given by:

$$\left(\frac{dT}{dt}\right)\bigg|_{t=2} = \left(\frac{\partial T}{\partial x}\right)\bigg|_{\vec{c}(2)} \left(\frac{dx}{dt}\right)\bigg|_{t=2} + \left(\frac{\partial T}{\partial y}\right)\bigg|_{\vec{c}(2)} \left(\frac{dy}{dt}\right)\bigg|_{t=2} + \left(\frac{\partial T}{\partial z}\right)\bigg|_{\vec{c}(2)} \left(\frac{dz}{dt}\right)\bigg|_{t=2}.$$

Let's compute: $\left(\frac{\partial T}{\partial x}\right)\bigg|_{\vec{c}(2)} = \left(-10e^{-x-2y-4z}\right)\bigg|_{\vec{c}(2)} = \boxed{-10\,e^{-31}}$

$$\left(\frac{\partial T}{\partial x}\right)\bigg|_{\vec{c}(2)} = \left(-20e^{-x-2y-4z}\right)\bigg|_{\vec{c}(2)} = \boxed{-20\,e^{-31}}$$

$$\left(\frac{\partial T}{\partial z}\right)\bigg|_{\vec{c}(2)} = \left(-40e^{-x-2y-4z}\right)\bigg|_{\vec{c}(2)} = \boxed{-40\,e^{-31}}$$

$\left(\frac{dx}{dt}\right)\bigg|_{t=2} = (2t)\bigg|_{t=2} = \boxed{4}$ $\qquad \left(\frac{dz}{dt}\right)\bigg|_{t=2} = \left(\frac{-1}{t^2}\right)\bigg|_{t=2} = \boxed{\frac{-1}{4}}$

$\left(\frac{dy}{dt}\right)\bigg|_{t=2} = (3t^2)\bigg|_{t=2} = \boxed{12}$

All together, we get

$$\left(\frac{dT}{dt}\right)\Big|_{t=2} = (-10e^{-31})(4) + (-20e^{-31})(12) + (-40e^{-31})\left(-\frac{1}{4}\right) = \boxed{-240e^{-31}}\ {}^{\circ}F/s$$

Where did this formula come from?

Notice we could rewrite $\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{dy}{dt}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{dz}{dt}\right)$ as

$$\nabla w(\vec{c}(t)) \bullet \vec{c}\,'(t) \quad \text{where} \quad \vec{c}(t) = (x(t), y(t), z(t))$$

Recall that ∇w and $\vec{c}\,'(t)$ are just special notations for the total derivatives $[Dw]$ and $[D\vec{c}]$.

So we could further rewrite the above equation as

$$[D(w \circ \vec{c})](t) = [Dw](\vec{c}(t))\,[D\vec{c}](t)$$

This is a 1x3 matrix

matrix mult. makes sense here

This is 3x1 matrix

The (total) derivative of $w \circ \vec{c}$ at t is the total derivative of w, evaluated at $\vec{c}(t)$ times the (total) derivative of $\vec{c}$ evaluated at t. This is essentially the same as in single variable, except now you multiply matrices!

More generally:

Theorem: (Chain Rule): Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^p \to \mathbb{R}^{n'}$ are both diff'ble. Then

$(f\circ g)(\vec{x}) = f(g(\vec{x}))$ is a fcn from $\mathbb{R}^p$ to $\mathbb{R}^n$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

If $\vec{x}$ is a p-dim. vector, $g(\vec{x})$ is n-dim. so it makes sense to plug it into f. $f(g(\vec{x}))$ is an m-dim. vector.

and $$[D(f\circ g)](\vec{x}) = [Df](g(\vec{x}))\,[Dg](\vec{x}).$$

$m\times p$ matrix $m\times n$ matrix $n\times p$ matrix

matrix mult. makes sense

<u>Example:</u> Let $g:\mathbb{R}^2\to\mathbb{R}^2$, $g(r,\theta)=(r\cos\theta, r\sin\theta)$
$f:\mathbb{R}^2\to\mathbb{R}^3$, $f(x,y)=(y^2, x^2y, x-y^2)$

Compute: ① $[Dg](2,\frac{\pi}{2})$
② $[Df](g(2,\frac{\pi}{2}))$
③ $[D(f\circ g)](2,\frac{\pi}{2})$.

① $$[Dg]=\begin{bmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta}\end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}$$

$$[Dg](2,\tfrac{\pi}{2}) = \begin{bmatrix} \cos(\frac{\pi}{2}) & -2\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & 2\cos(\frac{\pi}{2})\end{bmatrix} = \boxed{\begin{bmatrix} 0 & -2 \\ 1 & 0\end{bmatrix}}$$

② $$[Df]=\begin{bmatrix} 0 & 2y \\ 2x & 1 \\ 1 & -2y\end{bmatrix}$$

Notice: $g(2, \frac{\pi}{2}) = (2\cos(\frac{\pi}{2}), 2\sin(\frac{\pi}{2}))$
$= (0, 2)$

$$[Df](g(2,\tfrac{\pi}{2})) = [Df](0,2) = \boxed{\begin{bmatrix} 0 & 4 \\ 0 & 1 \\ 1 & -4 \end{bmatrix}}$$

③ $$[D(f\circ g)](2,\tfrac{\pi}{2}) = [Df](g(2,\tfrac{\pi}{2}))\,[Dg](2,\tfrac{\pi}{2})$$

$$= \begin{bmatrix} 0 & 4 \\ 0 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+(4)(1) & (0)(-2)+(4)(0) \\ (0)(0)+(1)(1) & (0)(-2)+(1)(0) \\ (1)(0)+(-4)(1) & (1)(-2)+(-4)(0) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & 0 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}}$$

Remark: If we write $(f\circ g)(r,\theta) = ((f\circ g)_1(r,\theta), (f\circ g)_2(r,\theta), (f\circ g)_3(r,\theta))$.
Then since

$$[D(f\circ g)](2,\pi/2) = \begin{bmatrix} 4 & 0 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}$$

we have, for example:

$$\left(\frac{\partial (f\circ g)_1}{\partial r}\right)(2,\tfrac{\pi}{2}) = 4$$

$$\left(\frac{\partial (f\circ g)_3}{\partial \theta}\right)(2,\tfrac{\pi}{2}) = -2$$

I.e., this matrix carries all the information about all the partial derivatives of $f\circ g$.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 10, 2017

Section:

- §2.5 (cont.)

Topics Covered:

- Chain rule for partial derivatives revisited.

Chain Rule for Partials: Last time we saw:

Theorem: (Chain Rule): Suppose $f: \mathbb{R}^m \to \mathbb{R}^p$, $g: \mathbb{R}^n \to \mathbb{R}^m$ are both diff'ble. Then

$$h := f \circ g : \mathbb{R}^n \to \mathbb{R}^p \text{ is also diff'ble}$$

and $$[Dh](\vec{x}) = [Df](g(\vec{x}))[Dg](\vec{x})$$

This means that we can obtain the full matrix of partial derivatives of $h = f \circ g$ by multiplying the matrices of partial derivatives of f and g. This is often more efficient than actually composing the functions and calculating the partials of h directly.

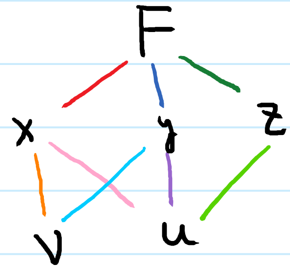
On the other hand, if we only need certain partials of h (and not the entire matrix $[Dh]$), we can do it in a quicker (and less confusing (?)) way.

Shortcut diagram for calculating partials of a composition:

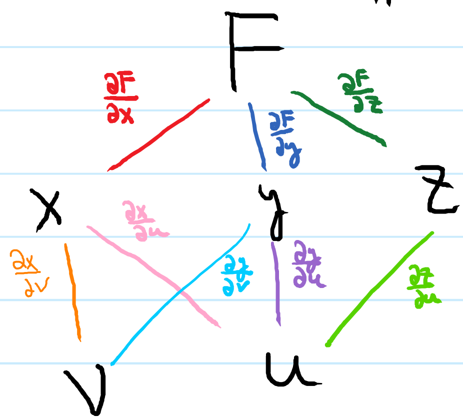
It is easiest to explain this via an example:

Suppose: ① $F(x,y,z)$ is a fcn of x, y, z,
② x, y are fcns of u, v
③ z is a fcn of only u.

Then we can draw a dependency diagram

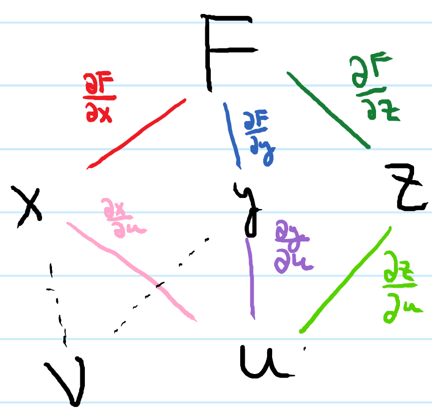


Here, we attach a line from one node to another if the higher node depends on the lower node. On this diagram, we attach the appropriate partial derivative:

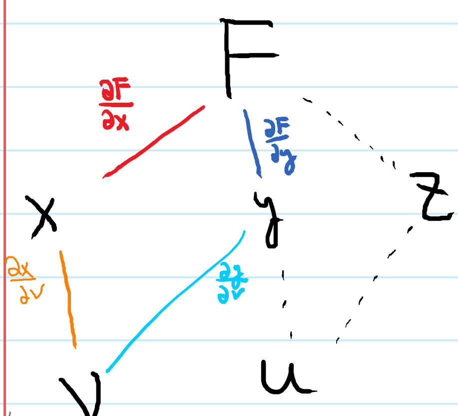


Now to find $\frac{\partial F}{\partial u}$, isolate 'u' in the tree and look at all paths from F to u.

On each path, multiply the partials on the legs. Finally, add the paths together:



$$\boxed{\frac{\partial F}{\partial u} = \left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial z}{\partial u}\right)}$$



If we do the same for v, we get:

$$\boxed{\frac{\partial F}{\partial v} = \left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right)}$$

Warning: To avoid messy notation, I didn't write where these fcns should be evaluated. Partials of f should be evaluated at $(x(u,v), y(u,v), z(u))$, partials of x, y, z, should be evaluated at (u,v).

Ex: In the example above, assume:

$x(1,2)=3 \qquad \frac{\partial F}{\partial x}(3,5,6)=8 \qquad \frac{\partial x}{\partial u}(1,2)=1 \qquad \frac{\partial y}{\partial u}(1,2)=-1$

$y(1,2)=5 \qquad \frac{\partial F}{\partial y}(3,5,6)=-1 \qquad \frac{\partial x}{\partial v}(1,2)=2 \qquad \frac{\partial y}{\partial u}(1,2)=5$

$z(1)=6 \qquad \frac{\partial F}{\partial z}(3,5,6)=7 \qquad \frac{\partial z}{\partial u}(1)=8$

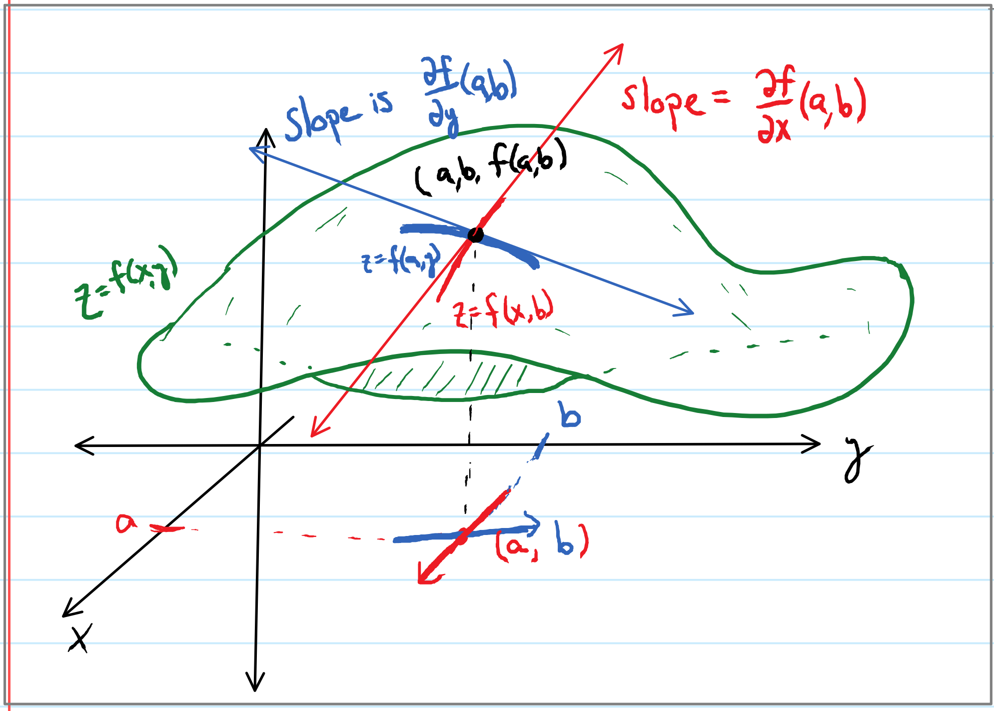
Then $\frac{\partial F}{\partial u}(1,2) = (3)(1)+(-1)(-1)+(7)(8)$
$= \boxed{60}$

Note: This notation means when $(u,v)=(1,2)$, $(x,y,z)=(3,5,6)$. So F's partials w.r.t. x,y,z should be evaluated at $(3,5,6)$.

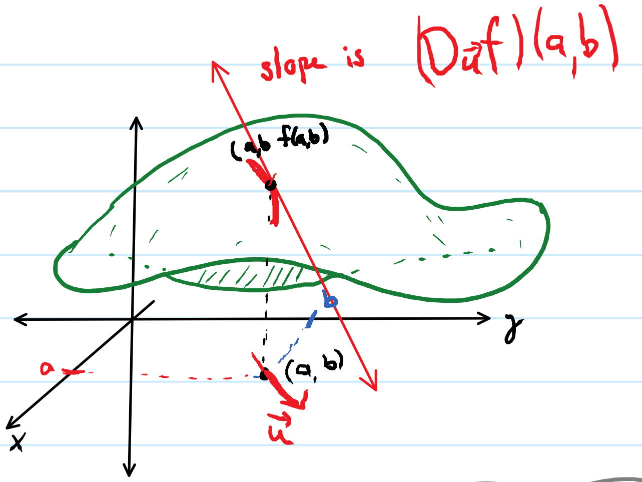
§2.6: Directional derivatives and the geometry of the gradient:

In this section, we stick to fcns from $\mathbb{R}^2\to\mathbb{R}$ or $\mathbb{R}^3\to\mathbb{R}$

Let $f:\mathbb{R}^2\to\mathbb{R}$. We saw that $\frac{\partial f}{\partial x}(a,b)$ (resp. $\frac{\partial f}{\partial y}(a,b)$) tell you the rate of change in f as we move from the point (a,b) in the positive x-direction (resp. positive y directn).



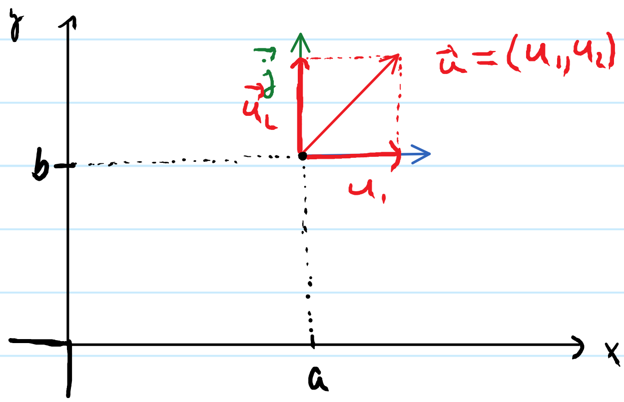
$\boxed{Q?}$ How do we determine the rate of change of f as we move from (a,b) in an arbitrary direction determined by a unit vector $\vec{u}$?



We will call this the directional derivative of f in the direction of $\vec{u}$, denoted $(D_{\vec{u}}f)(a,b)$

How do we calculate $(D_{\vec{u}}f)(a,b)$?

Look at the domain of f.



Decompose $\vec{u}$ into its components

$$\vec{u}=u_1\vec{i}+u_2\vec{j}$$

Since $\vec{u}$ "goes u_1 in the $\vec{i}$ and u_2 in the $\vec{j}$ direction, the total change in f as you move from (a,b) in the $\vec{u}$ direction is

$$(D_{\vec{u}}f)(a,b)=\underbrace{\frac{\partial f}{\partial x}(a,b)}_{\text{rate of change in } x \text{ dir.}}\ \underbrace{u_1}_{\text{distance travelled in } x \text{ dir.}}+\underbrace{\frac{\partial f}{\partial y}(a,b)}_{\text{rate of change in } y \text{ dir.}}\ \underbrace{u_2}_{\text{distance travelled in } y \text{ dir.}}$$

$$\boxed{=(\nabla f)(a,b)\cdot\vec{u}}$$

<u>Example:</u> Find the rate of change of $f(x,y,z)=2xy-e^{x^2y}+z^2$ at $(2,1,0)$ in the direction of $\vec{u}=(1,-1,1)$.

(1) According to my definition, we need $\vec{u}$ to be a unit vector, so we first normalize $\vec{u}$.

Replace $\vec{u}$ by $\vec{e}_{\vec{u}} = \frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{1+1+1}}(1,-1,1)$

$$= \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

(2) The equation above was for fcns from $\mathbb{R}^2$ to $\mathbb{R}^1$, but the same formula works.

$$(\nabla f)(2,1,0) = \left(2y - 2xye^{x^2y},\ 2x - x^2e^{xy},\ 2z\right)\Big|_{(2,1,0)}$$

$$= (2-4e^4,\ 4-4e^4,\ 0)$$

$$(D_{\vec{u}}f)(2,1,0) = (2-4e^4,\ 4-4e^4,\ 0)\cdot\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}(2-4e^4) - \frac{1}{\sqrt{3}}(4-4e^4)$$

$$= \boxed{\frac{-2}{\sqrt{3}}}$$

Differences with the book:

Warnings: (1) Your book also considers $D_{\vec{u}}f$ when $\vec{u}$ is not a unit vector. They call this the directional derivative along $\vec{u}$. We will only look at directional derivatives in the direction of $\vec{u}$.

(2) The book uses the notation $[Df](a,b)(\vec{u})$ instead of $(D_{\vec{u}}f)(a,b)$.

This is because $(D_{\vec{u}}f)(a,b) = (\nabla f)(a,b)\cdot\vec{u}$

$$= [Df](a,b)\cdot\vec{u}$$

the total derivative of f

viewed as a column vector.

③ The book has a nice explanation as to what this means in higher dimension. We will ignore this, but I encourage you to read it since it may provide insight for the significance of $[Df](\vec{x})$.

Geometry of the gradient: If $\vec{u}$ is a unit vector,

$(D_{\vec{u}} f)(a,b) = (\nabla f)(a,b) \cdot \vec{u}$ is the rate of change in the $\vec{u}$ direction.

On the other hand:

$$(\nabla f)(a,b) \cdot \vec{u} = \|(\nabla f)(a,b)\| \cdot \|\vec{u}\| \cos\theta$$
$$= \|\nabla f(a,b)\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$.

Note: ① $(D_{\vec{u}} f)(a,b)$ is maximal when $\cos\theta = 1$
$\Leftrightarrow \theta = 0$
$\Leftrightarrow \vec{u}$ points in the same direction as $(\nabla f)(a,b)$

So $(\nabla f)(a,b)$ points in the direction of maximal increase for f.

② If $\vec{u} = \dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$ is the unit vector in the $\nabla f(a,b)$ direction, $(D_{\vec{u}} f)(a,b) = \nabla f(a,b) \cdot \left(\dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|} \right)$

$$= \frac{\nabla f(a,b) \cdot \nabla f(a,b)}{\|\nabla f(a,b)\|}$$

$$= \frac{\|\nabla f(a,b)\|^2}{\|\nabla f(a,b)\|}$$

$= \|\nabla f(a,b)\|$

So the maximal rate of change of f in any direction is $\|\nabla f(a,b)\|$

<u>Example</u>: Farmer Corey is standing near the middle of his corn field in the great city of Omaha, NE (say he is standing at the point $(0,1)$). The temperature at the point (x,y) is given by $T(x,y) = 30e^{-(x-1)^2-(2y-1)^2}$. Farmer Corey is very cold! What direction should he walk to warm up the fastest?

<u>Sol</u>: He should walk in the direction of $(\nabla T)(0,1)$.

$$\nabla T = \left(-60(x-1)e^{-(x-1)^2-(2y-1)^2},\ -120(2y-1)e^{-(x-1)^2-(2y-1)^2} \right)$$

$$(\nabla T)(0,1) = \left(60e^{-2},\ -120e^{-2} \right).$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 13, 2017

Section:
- [ ] § 2.6 (cont.)

Topics Covered:
- [ ] More geometry of the gradient
- [ ] Equations of tangent planes to surfaces

From last time: If $f:\mathbb{R}^n \to \mathbb{R}$ (usually $n=2,3$) and if $\vec{u}$ is a unit vector, the directional derivative of f at $\vec{x}_0$ in the direction of $\vec{u}$ is:

$$\boxed{\begin{aligned}(D_{\vec{u}}f)(\vec{x}) &= \lim_{t\to 0} \frac{f(\vec{x}+t\vec{u})-f(\vec{x})}{t} \\ &= (\nabla f)(\vec{x})\cdot\vec{u}.\end{aligned}}$$

$(D_{\vec{u}}f)(\vec{x})$ tells us the rate of change in f in the direction of $\vec{u}$. Using the identity

$(\nabla f)(\vec{x})\cdot\vec{u} = \|\nabla f(\vec{x})\|\|\vec{u}\|\cos\theta$ where θ is the angle between $\nabla f(\vec{x})$ and $\vec{u}$, we saw:

1. $(\nabla f)(\vec{x})$ points in the direction of steepest increase,
2. The maximal rate of increase of f at $\vec{x}$ is $\|\nabla f(\vec{x})\|$.

Example: Farmer Corey is standing near the middle of his corn field in the great city of Omaha, NE (say he is standing at the point $(0,1)$). The temperature at the point (x,y) is given by $T(x,y) = 30e^{-(x-1)^2-(2y-1)^2}$. very cold! What direction should he walk to warm up the fastest?

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Sol: He should walk in the direction of $(\nabla T)(0,1)$.

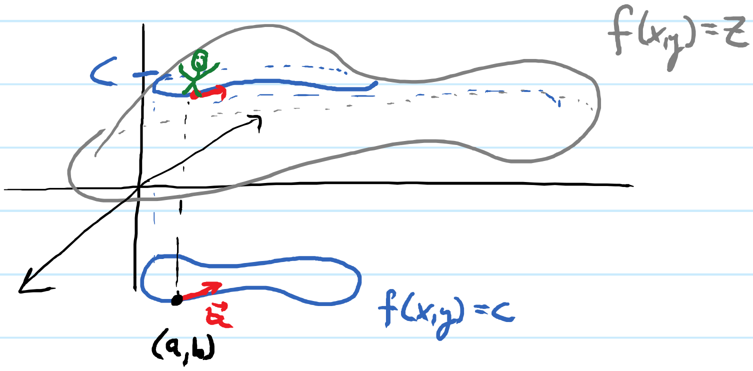
$$\nabla T = \left(-60(x-1)e^{-(x-1)^2-(2y-1)^2},\ -120(2y-1)e^{-(x-1)^2-(2y-1)^2}\right)$$

$$(\nabla T)(0,1) = \left(60e^{-2},\ -120e^{-2}\right).$$

Level sets revisited: If $f:\mathbb{R}^n \to \mathbb{R}$, and c is a constant, the level set of height c is the set of points $(x_1, \dots, x_n)$ that satisfy $f(x_1, .., x_n) = c$.

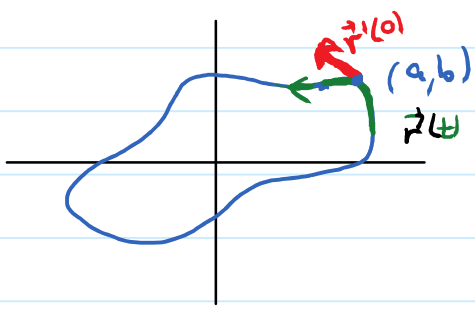
Suppose now that $f:\mathbb{R}^2 \to \mathbb{R}$, and that $\vec{u}$ is a unit vector that is tangent to the level curve $f(x,y)=c$.

If you walk on the graph of f along the level curve $f(x,y)=c$, you will not be ascending nor descending since your height will always be c.



Therefore, we should expect $(D_{\vec{u}} f)(a,b) = 0$.

Let's prove this: Let $\vec{r}(t)$ be a parametrization of the level curve near (a,b), such that $\vec{r}(0) = (a,b)$ and $\vec{r}\,'(0) = \vec{u}$



Since f is constant on the level curve, and since $\vec{r}(t)$ is a point on the level curve for every t, we have $f(\vec{r}(t)) = c$. But then

$$\frac{d}{dt}(f \circ \vec{r})(0) = \frac{d}{dt}(c)$$

$$\Rightarrow \quad \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0$$

$$\Rightarrow \quad \nabla f(a,b) \cdot \vec{u} = 0$$

but then $(D_{\vec{u}} f)(a,b) = \nabla f(a,b) \cdot \vec{u} = 0$, which is what we wanted.

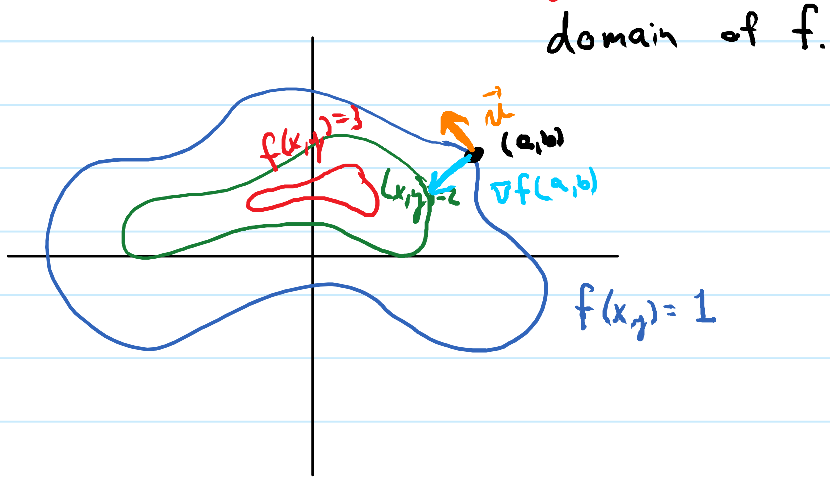
Remark: As a corollary to the above argument, we see that if $\vec{u}$ is a unit vector that is tangent to the level curve at (a,b), then

$$\nabla f(a,b) \cdot \vec{u} = 0$$
$$\iff \nabla f(a,b) \perp \vec{u}$$

This tells us that:

The gradient vector is orthogonal to level sets!

Ex:

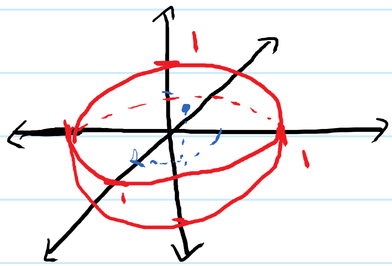


Rate of change of f is 0 in the $\vec{u}$ direction and is maximal in the direction of $\nabla f(a,b)$.

Application: Tangent planes to surfaces:

Problem: Find the equation of the plane that is tangent to the sphere $x^2+y^2+z^2=1$ at the point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

picture:



Idea: We could try to solve for z in terms of (x,y). This way we write z as a fcn of x and y, and we can use our old formula for tangent planes. However, $x^2+y^2+z^2=1$

$$\Rightarrow \quad z = \pm\sqrt{1-x^2-y^2}.$$

The $\pm$ tells us z is not a fcn of x and y, so we try something different:

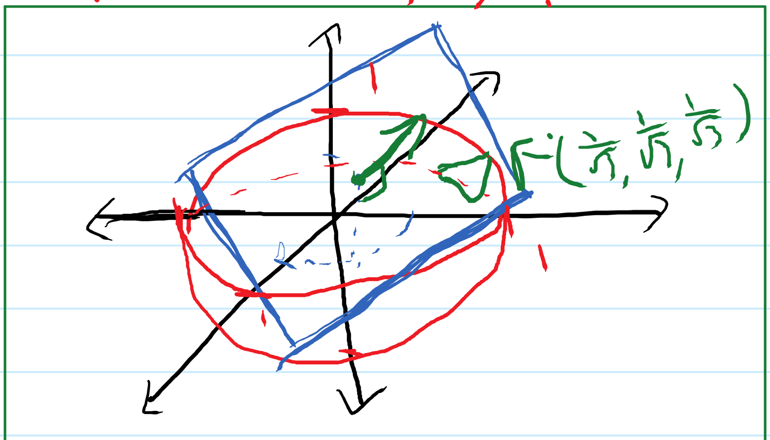
Let $F(x,y,z) = x^2+y^2+z^2$. Then the sphere $x^2+y^2+z^2=1$, is just the level surface of F with height 1:

$$F(x,y,z)=1 \iff x^2+y^2+z^2=1.$$

Since gradients are orthogonal to level sets,

$(\nabla F)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is orthogonal to the sphere $x^2+y^2+z^2=1$.

In particular, $(\nabla F)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is normal to the tangent plane at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.



$$\nabla F = (2x, 2y, 2z).$$

$$\boxed{\nabla F\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) = \left(\tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}\right)}$$

Now we have a normal vector:

$\vec{N} = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, and a point on the plane $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

So we can write the equation of the plane:

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$$

$$\boxed{\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = 2}$$

Note: To make this look nicer, we could multiply by $\frac{\sqrt{3}}{2}$ to get:

$$\boxed{x + y + z = \sqrt{3}}$$

Example: Find an equation to the tangent plane to the surface

$$e^{y^2} z^4 x - x^4 z = 0 \quad \text{at} \quad (1, 0, 1).$$

Note: The set of points that satisfy this equation forms some surface in $\mathbb{R}^3$. I am not able to draw it, and I certainly can't solve for z to get a fcn of x and y. Let's use the same strategy as last time:

Let $F(x,y,z) = e^{y^2} z^4 x - x^4 z$. Then the surface $e^{y^2} z^4 x - x^4 z = 0$ is the level surface $F(x,y,z) = 0$.

Therefore, $\nabla F(1,0,1)$ is $\perp$ to the surface at $(1,0,1)$ and hence $\nabla F(1,0,1)$ is normal to the tangent plane.

$$\nabla F = \left(e^{y^2} z^4 - 4x^3 z,\ 2y e^{y^2} z^4 x,\ 4 e^{y^2} z^3 x - x^4\right)$$

$$\boxed{\nabla F(1,0,1) = (-3, 0, 3)}$$

Now we have our normal vector: $\vec{N} = (-3, 0, 3)$, and a point on the plane: $(1, 0, 1)$.

So we get the plane:

$$-3x + 0 \cdot y + 3z = (-3, 0, 3) \cdot (1, 0, 1)$$

$$\Rightarrow \quad \boxed{-3x + 3z = 0}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 15, 2017

Section:

- [ ] Equations of tangent planes via implicit differentiation (not in the book)
- [ ] § 3.1

Topics Covered:

- [ ] Implicit differentiation
- [ ] Iterated partial derivatives and Clairaut's Theorem on the equality of mixed partials

Last time: Given a surface defined by some equation $F(x,y,z)=c$ (for example, $x^2+y^2+z^2=3$) we saw that the vector $\nabla F(a,b,c)$ is orthogonal to the surface $F(x,y,z)=c$ (because gradients are $\perp$ to level sets!)
This gave us a way to calculate eqns of tangent planes to arbitrary surfaces.

Another option (that is not mentioned in the book as far as I know) is to use implicit differentiation.

Example from before: Find the eqn of the plane tangent to the sphere $x^2+y^2+z^2=3$ at $(1,1,1)$ using implicit differentiation.

Idea: We saw last time that we cannot solve for z, and express z as a fcn of x,y. In this case, we say z is not an explicit fcn of x,y.
Nevertheless, since $x^2+y^2+z^2=3$, there is a clear restriction for what z can be depending on x and y.
So we say z is implicitly a fcn of x and y.
It therefore makes sense to compute $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.
We can then use the "old" tan. plane formula:

$$(*) \quad z = c + \frac{\partial z}{\partial x}(a,b)(x-a) + \frac{\partial z}{\partial y}(a,b)(y-b)$$

where (a,b,c) is the point of tangency.

Q? How do we find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

Ans: Use implicit differentiation on the equation $x^2+y^2+z^2=3$.

Take the partial w.r.t x: (differentiate the eqn w.r.t. x: we still pretend y is constant, but when we see z, we diff. wrt. z, then multiply by $\frac{\partial z}{\partial x}$. This is just like implicit diff. in calc. 1, except y is const. Note, this process works because of the chain rule).

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial x}(3)$$

$$2x + \underbrace{2z \cdot \frac{\partial z}{\partial x}}_{\text{Chain rule}} = 0$$

$$\Rightarrow 2z\frac{\partial z}{\partial x} = -2x$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-x}{z}}$$

Take partial w.r.t y: (This time x is const.)

$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial y}(3)$$

$$\Rightarrow 2y + 2z \cdot \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-y}{z}}$$

Since the point we are looking at is $P_0 = (1,1,1)$, we have

$$\boxed{\begin{aligned}\left(\frac{\partial z}{\partial x}\right) &= \frac{-1}{1} = -1\\ \frac{\partial z}{\partial y} &= \frac{-1}{1} = -1\end{aligned}}$$

Now using eqn $(*)$, the tan. plane is given by

$$z = 1 - 1(x-1) - 1(y-1)$$

$\Rightarrow$ $\boxed{z + x + y = 3}$ which is the same as before.

Remark: ① This process doesn't always work: sometimes you are forced to have 0 in the denominator.

② We are really doing the same thing as before but wording it differently.

§3.1: Iterated partial derivatives:

The next major goal in this class will be to optimize fcns of two variables. Meaning we want to:

① Find local/global max/mins.

② Find critical points (points where the tan plane is horizontal)

③ Classify crit. pts. as local max / local min / other by using concavity. In calc. I, this is called the second derivative test.

We start by addressing goal ③ and discussing the multivariable version of second derivatives:

Iterated Partial Derivatives:

Start with a fcn $f(x,y)$. Then we can take partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$. Now

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are themselves functions of x and y, so it makes sense to take partial derivatives of $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ w.r.t. x, y. The result will be called a second order partial derivative.

Notation: ① $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ (or f_{xx}) means take the x partial twice.

② $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ (or f_{xy}) means take x-partial then the y-partial.

③ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ means take the y partial twice.

④ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ (or f_{yx}) means take y-partial, then the x-partial

We will talk about what these represent soon but first:

Examples: Find ALL second order partials for:

① $f(x,y) = e^{-y}\ln(x) - x^2y^2$.

$$\frac{\partial f}{\partial x} = \frac{e^{-y}}{x} - 2xy^2$$

$$\frac{\partial f}{\partial y} = -e^{-y}\ln(x) - 2x^2y$$

$$\boxed{\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{-e^{-y}}{x^2} - 2y^2 \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{-e^{-y}}{x} - 4xy \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{-e^{-y}}{x} - 4xy \\ \frac{\partial^2 f}{\partial y^2} &= e^{-y}\ln(x) - 2x^2 \end{aligned}}$$

② $g(x,y) = y^3x - x^2y^2 + 4x^2y$

$$\frac{\partial g}{\partial x} = y^3 - 2xy^2 + 8xy$$

$$\frac{\partial^2 g}{\partial x^2} = 2y^2 + 8y$$

$$\frac{\partial^2 g}{\partial y \partial x} = 3y^2 - 4xy + 8x$$

$$\frac{\partial g}{\partial y} = 3y^2x - 2x^2y + 4x^2$$

$$\frac{\partial^2 g}{\partial x \partial y} = 3y^2 - 4xy + 8x$$

$$\frac{\partial^2 g}{\partial y^2} = 6yx - 2x^2$$

③ $h(x,y) = \cos(xy) + y^2$

$\frac{\partial h}{\partial x} = -\sin(xy)\cdot y$

$\frac{\partial h}{\partial y} = -x\sin(xy) + 2y$

$\frac{\partial^2 h}{\partial x^2} = -y^2\cos(xy)$

$\frac{\partial^2 h}{\partial y \partial x} = -\sin(xy) + y(-x\cos(xy))$
$= -\sin(xy) - xy\cos(xy)$

$\frac{\partial^2 h}{\partial x \partial y} = -\sin(xy) - x(\cos(xy)\cdot y)$
$= -\sin(xy) - xy\cos(xy)$

$\frac{\partial^2 h}{\partial y^2} = -x^2\cos(xy) + 2$

In each of these examples, we see that the mixed partials $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ are equal!

It turns out that if the fcns are "nice enough", this will always be true.

Theorem: (Clairaut): If the mixed partials of $f(x,y)$ and are continuous, then they are equal:

$$\frac{\partial^2 f}{\partial xy} = \frac{\partial^2 f}{\partial yx}$$

Rmk: This is a great way to check your work!

Example: Does there exist a fcn with continuous 2^{nd} order partials such that:

$\frac{\partial f}{\partial x} = 4x^2 + y$ $\qquad$ $\frac{\partial f}{\partial x} = 4y^3x + x.$

Sol: If there were, then

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \text{while} \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y^3 + 1$$

By Clairaut's thm, this is impossible.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 17, 2017

Section:

- §3.3

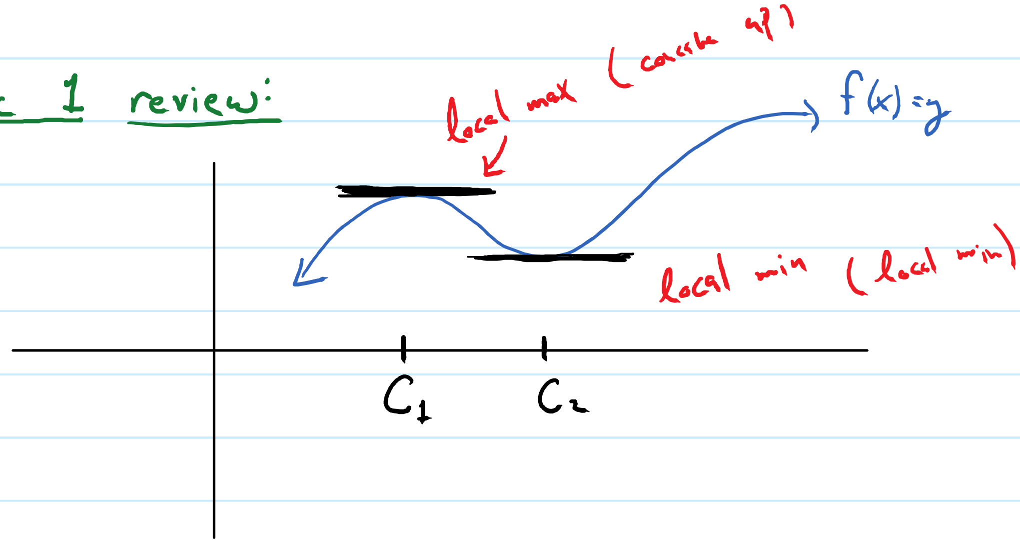
Topics Covered:

- Review of local maxima and minima for functions of one variable
- Local maxima, local minima, saddle points for functions of two variables

§3.2 Part 1: Optimization - Local extrema and saddle points:

Def: A fcn f (of 1, 2, or 3 input variables for us) has a local maximum at x_0 if for all other x near x_0, $f(x_0) \geq f(x)$.
The notion of local minimum is defined similarly.

Quick Calc 1 review:



A cont. fcn, f, potentially has local extrema at critical points:

I.e., points where $f'(x)=0$ (horizontal tangent line) or where f' is undefined.

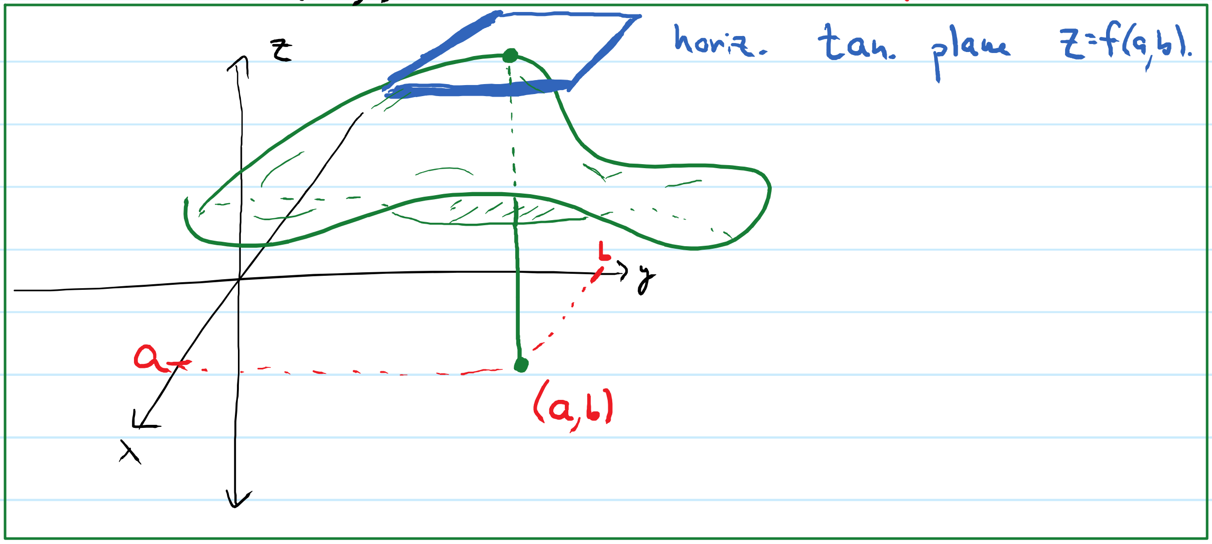
We can test if a critical point, a, gives a local max (resp. local min) by checking if $f''(a)>0$, which means the graph is concave down at a (resp. $f''(a)<0$, which means the graph is concave up at a).

Local extrema for fcns $z=f(x,y)$:

A fcn $f(x,y)=z$ potentially has a local max. or local min at points (a,b) where the tangent plane: $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ is horizontal. It is not hard to see that this happens

(or is undefined)

iff both $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$,
i.e., when $\nabla f(a,b)=0$. (or when either partial is undefined.



Example: ① Find all crit. pts of $f(x,y)=x^2-3xy+5x-2y+6y^2+8$.

Sol: $\nabla f=(2x-3y+5,\ -3x-2+12y)$
Both partials are cont. everywhere so we must solve the system of equations:

$$\left.\begin{array}{ll}\frac{\partial f}{\partial x}: & 2x-3y+5=0\\ \frac{\partial f}{\partial y}: & -3x-2+12y=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}2x-3y=-5\\ -3x+12y=2\end{array}\right\} \Rightarrow \boxed{3y=2x+5}$$

plug into eqn 2:

$$-3x+4(3y)=2$$
$$-3x+4(2x+5)=2$$
$$\Rightarrow 5x+20=2$$
$$\Rightarrow \boxed{x=\frac{-18}{5}} \Rightarrow y=\frac{1}{3}\left(2\left(\frac{-18}{5}\right)+5\right)$$
$$y=\frac{1}{3}\left(\frac{-11}{5}\right)$$
$$\boxed{y=\frac{-11}{15}}$$

so there is one crit. pt. at $\left(\frac{-18}{5},\frac{-11}{15}\right)$

② $g(x,y)=8y^3x+12x^2-24xy$.

p

Sol g is a polynomial and is therefore diff'ble everywhere

So we try to solve: $\nabla g = 0$

$\nabla g = (8y^2+24x-24y,\ 16yx-24x) = 0$. So we solve the system:

$$\boxed{\begin{array}{ll} \text{I} & 8y^2+24x-24y=0 \\ \text{II} & 16yx-24x \overset{?}{=} 0 \end{array}}$$

II $\Rightarrow$ $8x(2y-3)=0$ $\Rightarrow$ either $\boxed{x=0}$ or $\boxed{y=\frac{3}{2}}$ (possibly both).

Case 1: If $x=0$, I $\Rightarrow$ $8y^2-24y=0$

$\Rightarrow 8y(y-3)=0$

$\Rightarrow \boxed{y=0}$ or $\boxed{y=3}$

So from case 1: we get two crit. pts $\boxed{(0,0)}$, $\boxed{(0,3)}$

Case 2: If $y=\frac{3}{2}$, I $\Rightarrow$ $8\left(\frac{3}{2}\right)^2+24x-24\left(\frac{3}{2}\right)=0$

$\Rightarrow 8\left(\frac{9}{4}\right)+24x-36=0$

$\Rightarrow 18+24x-36=0$

$\Rightarrow 24x=18$

$\Rightarrow \boxed{x=\frac{3}{4}}$

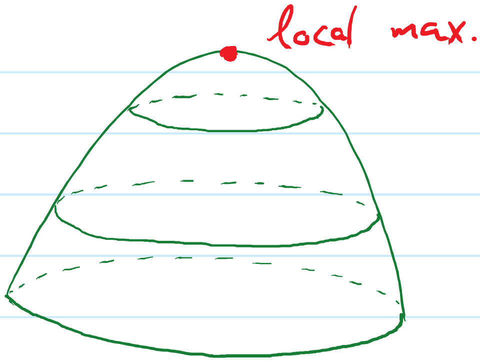
So case 2 gives us a crit. pt. $\boxed{\left(\frac{3}{4},\frac{3}{4}\right)}$

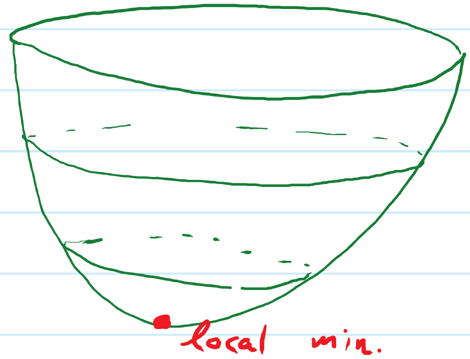
So g has 3 crit. pts: $\boxed{(0,0),(0,3),\left(\frac{3}{4},\frac{3}{4}\right)}$.

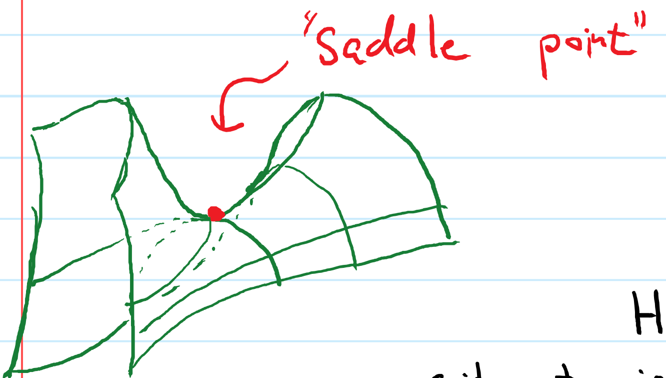
Classification of critical points:

(has cont. 2nd partials)

For "nice" fns $f(x,y)$ of 2 variables, there are three types of critical points:







If we zoom in on the graph of f at a crit. pt., it looks similar to one of these.

How do we know if a crit. pt. is a local max./min./saddle?.

Thm: (Second derivative test): Let $f(x,y)$ be a fcn with cont. 2^{nd} order partials. Let (a,b) be a crit. pt. of f. Let

$$D(a,b) = \left(\left(\frac{\partial^2 f}{\partial x^2}\right)(a,b)\right)\left(\frac{\partial^2 f}{\partial y^2}(a,b)\right) - \left(\left(\frac{\partial^2 f}{\partial x \partial y}\right)(a,b)\right)^2.$$

D is called the discriminant at (a,b). Then

(I) If $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, f has a local min at (a,b).

(II) If $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, f has a local max at (a,b).

(III) If $D(a,b) < 0$, f has a saddle point at (a,b).

Rmk: (1) If $D(a,b) = 0$, the test does not give us any information!

(2) $D(x,y)$ is the determinant of the Hessian matrix

of f: $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$

③ A proper explanation of the test would require a discussion of multivariable Taylor Polynomials, and a development of some linear algebra that is beyond the scope of math 18, so we will skip it. See the book for details.

Example: Classify the critical points: $(0,0), (0,3), (\frac{45}{48}, \frac{3}{4})$ for the fcn $g(x,y) = 8y^2x + 12x^2 - 24xy$.

Sol: Recall $\frac{\partial f}{\partial x} = 8y^2 + 24x - 24y$ $\qquad$ $\frac{\partial f}{\partial y} = 16yx - 24x$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 24 \qquad\qquad \frac{\partial^2 f}{\partial y^2} = 16x$$

$$\frac{\partial^2 f}{\partial y \partial x} = 16y - 24 \quad = \quad \frac{\partial^2 f}{\partial x \partial y} = 16y - 24$$

Therefore, $D(x,y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial f}{\partial y \partial x}\right)^2$
$= 24(16x) - (16y - 24)^2.$

For $(0,0)$: $D(0,0) = 24(16(0)) - (-24)^2 < 0$

Therefore $(0,0)$ is a saddle point.

For $(0,3)$: $D(0,3) = 24(16(0)) - (16(3) - 24)^2$
$= 0 - (16(3) - 24)^2 < 0$

Therefore $(0,3)$ is a saddle point.

For $(\frac{3}{4}, \frac{3}{4})$:

$$D\left(\tfrac{3}{4},\tfrac{3}{4}\right) = (24)(16)\left(\tfrac{3}{4}\right) - \left(16\left(\tfrac{3}{4}\right) - 24\right)^2$$
$$= (24)(12) - (12-24)^2$$
$$= 2(12^2) - (-12)^2 > 0.$$

$\left(\frac{3}{4},\frac{3}{4}\right)$ may be a local max. or min. So let's check

$\frac{\partial^2 f}{\partial x^2}\left(\frac{3}{4},\frac{3}{4}\right) = 24 > 0.$ Therefore, f has a local min. at $\left(\frac{3}{4},\frac{3}{4}\right)$.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 22, 2017

Section:

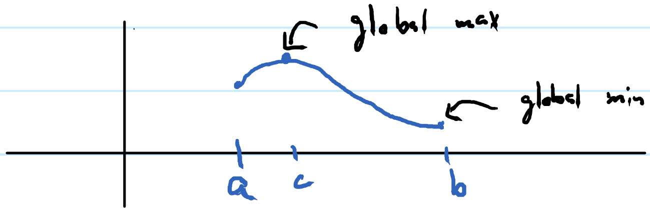
- [ ] § 3.3 (cont.)

Topics Covered:

- [ ] Closed sets
- [ ] Extreme Value Theorem
- [ ] Global Extrema

§3.3 (cont.):

Recall: If $f(x)$ is a continuous fcn on a closed interval $[a,b]$, then f has a global max. and global min in the interval $[a,b]$. Moreover, the global extrema occur either at critical pts. or the endpts.

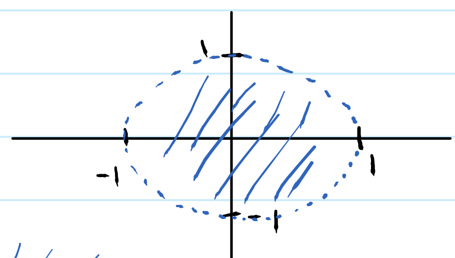


If the domain of a fcn is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ what should we mean by "endpoints".

Informal definitions: ① A subset, $\mathcal{U}$, of $\mathbb{R}^n$ is called bounded if $\mathcal{U}$ "does not go off to ∞".

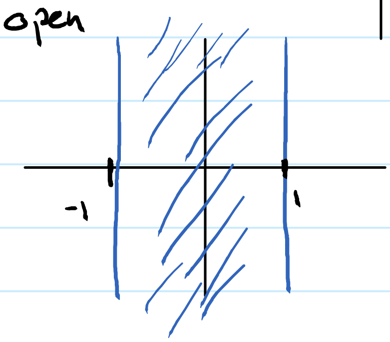
② A subset, $\mathcal{U}$, of $\mathbb{R}^n$ is called closed if $\mathcal{U}$ contains its boundary.

Examples: ① $\{(x,y) \mid x^2+y^2<1\}$



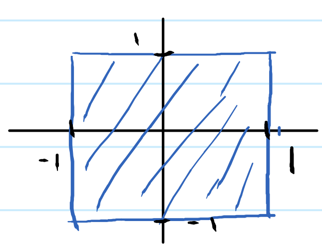
bounded, but not open

② $\{(x,y) \mid -1 \le x \le 1\}$



Includes its boundary, but goes off to ∞.
This set is closed, but not bounded.

③ $\{(x,y) \mid -1 \le x \le 1,\ -1 \le y \le 1\}$



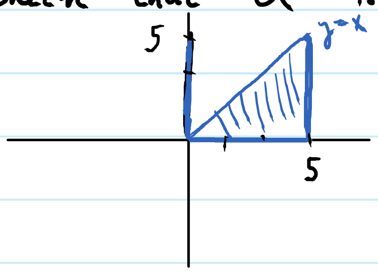
This set is closed & bounded.

Rmk: A subset, $\mathcal{U}$, of $\mathbb{R}^n$ that is closed and bounded is called compact.

Thm: (Extreme Value Theorem): If f is a cont. fcn defined on a closed and bounded subset $\mathcal{U}$ of $\mathbb{R}^n$ (for us $n=2$ or 3) then f has a global max and a global min. on $\mathcal{U}$. Moreover, the global extrema must occur at critical points, or on the boundary, $\partial\mathcal{U}$, of $\mathcal{U}$.

Examples: ① Find global extrema for the fcn
$$f(x,y) = x^2y - x - 3xy^2$$
on the square $\mathcal{U} = \{(x,y) \mid 0 \le x \le 5, 0 \le y \le x\}$

Sol: Check that $\mathcal{U}$ is closed & bounded.



yes, $\mathcal{U}$ is closed and bounded.

Step 1: Find critical points on the interior.

Since f is a polynomial, it is diff'ble everywhere, so we must solve $\vec{\nabla} f = \vec{0}$.
$$\vec{\nabla} f = (2xy - 1 - 3y^2,\ x^2 - 6xy) = \vec{0}$$
$$\Leftrightarrow \quad \text{I} \quad 2xy - 1 - 3y^2 = 0$$
$$\text{II} \quad x^2 - 6xy = 0$$

II $\Rightarrow$ $x(x-6y) = 0 \Rightarrow$ either $x=0$ or $x=6y$

Since the only point where $x=0$ is on the boundary, (which we will check separately) we can ignore the case where $x=0$.

So we get $\boxed{x=6y}$. Plug this back into eqn I

$$2(6y)y - 1 - 3y^2 = 12y^2 - 1 - 3y^2 = 0 \Rightarrow 9y^2 = 1 \Rightarrow$$

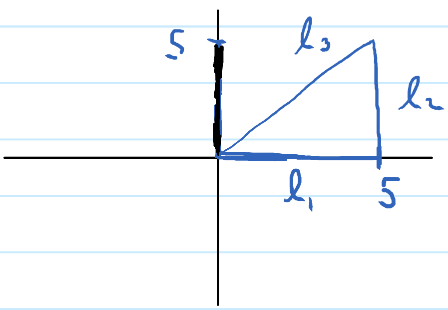
$\boxed{y = \pm\frac{1}{3}}$, Since every point in $\mathcal{U}$ has a positive y-value, $y = -\frac{1}{3}$ is not in the domain, so we can disgard it, and we get $\boxed{y = \frac{1}{3}}$. Since $x = 6y$,

$\boxed{x = 2}$

So we have one critical pt. on the interior of $\mathcal{U}$:

$\boxed{(2, \frac{1}{3})}$

Step 2: Find potential max/mins along the boundary.



Note that the boundary can be split into three line segments that are easy to parametrize. We must look for potential extrema on all three sides.

ℓ_1: ℓ_1 can be described as: $y = 0$, and $0 \le x \le 5$.

So the fcn f restricted to ℓ_1 becomes

$$f_1(x) = f(x, 0) = -x \quad , \quad 0 \le x \le 5.$$

This is a fcn of one variable defined on a closed interval. it's max./mins. happen at crit. pts. or the endpts.

$f_1'(x) = -1 \rightarrow$ no crit. pts., so we only need the endpts. $x = 0$, $x = 5$. Since $y = 0$ on ℓ_1, we pick up two more points that could potentially be global extrema: $\boxed{(0,0)}$, $\boxed{(5,0)}$

$\underline{l_2}$: l_2 can be described by: $x=5$, $0\le y\le 5$. So f restricted to l_2 becomes

$$f_2(y)=f(5,y)=25y-5-15y^2\ ,\qquad 0\le y\le 5.$$

Again, this is a one variable fcn, so we use calc 1 techniques.

Find crit. pts. for f_2: $f_2'(y)=-30+25y=0 \Rightarrow \boxed{y=\frac{6}{5}}$

We must also consider the end pts. $y=0$, $y=5$.

Since $x=5$ on l_2, we get two new points that could potentially be max./mins. for f: $\boxed{(5,\frac{6}{5}),(5,5).}$

Lastly,

$\underline{l_3}$: l_3 can be described as: $y=x$, $0\le x\le 5$. So f restricted to l_3 becomes:

$$f_3(x)=f(x,x)=x^3-x-3x^3$$
$$=-2x^3-x\qquad 0\le x\le 5$$

which is a fcn of one variable.

Find crit. pts. for f_3: $f_3'(x)=-6x^2-1=0$
$\Rightarrow x^2=\frac{-1}{6}$ which is impossible.

So we only need to look at the endpts:
$x=0$, $x=5$.

On l_3, $y=x$ so our potential points on l_3 are $(0,0)$, $(5,5)$, which we already had.

$\underline{\text{Step 3:}}$ Plug in all of the potential points to see which gives us a global max, and which gives a global min.

Our potential points are: $(2,\frac{1}{3}),(0,0),(5,0),(5,5),(5,\frac{6}{5})$.

plug each into $f(x,y) = x^2y - x - 3xy^2$:

$f(0,0) = 0$

$f(5,0) = -5$

$f(5,5) = 5^2 \cdot 5 - 5 - 3 \cdot 5^2 \cdot 5 = -255$ smallest

$f(2,\frac{1}{3}) = \frac{4}{3} - 2 - \frac{6}{9} = \frac{4}{3} - \frac{6}{3} - \frac{2}{3} = -\frac{4}{3}$

$f(5,\frac{6}{5}) = 25\left(\frac{6}{5}\right) - 5 - 3(5)\left(\frac{36}{25}\right)$

$= 30 - 5 - \frac{108}{5}$

$= \frac{17}{5}$ largest.

So f has a global max. of $\frac{17}{5}$ at $(5,\frac{6}{5})$ and
f has a global min. of -255 at $(5,5)$.

Summary: Step 1: Find critical points on the interior of U.

Step 2: Ⓐ Find a way to describe the boundary so that f restricted to the boundary becomes a fcn of 1-variable.

Ⓑ Find all potential points on the boundary using Calc 1.

Step 3: Plug all of the points you found in step 1, 2 into f to find the global m

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 24, 2017

Section:

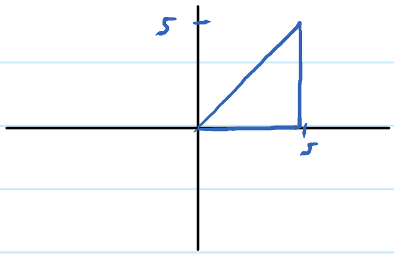
- [ ] §3.4

Topics Covered:

- [ ] Optimization using the method of Lagrange Multipliers

§3.4: The method of Lagrange multipliers

Last time, we saw how to find extreme values of a cont. fcn f, constrained to the triangle



We did this by finding a nice parametrization of the sides of the triangle and then composing with f. Then we turned the problem into a calc 1 problem. If the constraint is hard to parametrize, then this techique isn't ideal.

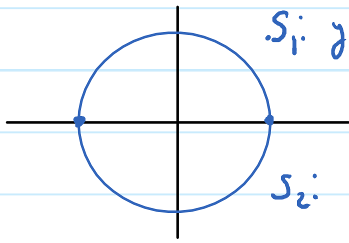
Example: Say we want to find the max./min. of the fcn $f(x,y) = xy$ subject to the constraint $x^2+y^2=1$.

I.e., we want to find max./min. of f restricted to the domain $C = \{(x,y) \mid x^2+y^2=1\}$.

Let's first note that f is continuous and the domain is the unit circle, which is closed and bounded. So by the extreme value thm, f does have global max./mins.

Second, note that the graph of f is the saddle, so we should already expect max at $(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}), (\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$ and min at $(\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$, $(\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$.

Using techiques from last time, we can split the circle into an upper hemisphere and lower hemisphere, each of which can be parametrized:



S_1: $y = \sqrt{1-x^2}$, $-1 \le x \le 1$

S_2: $y = -\sqrt{1-x^2}$, $-1 \le x \le 1$

$\underline{S_1}$: f restricted to S_1 becomes

$$f_1(x) = f(x, -\sqrt{1-x^2}) = x\sqrt{1-x^2} \quad . \quad -1 \le x \le 1.$$

$\underline{\text{Find critical pts:}}$ $f_1'(x) = \sqrt{1-x^2} + x\left(\frac{-x}{\sqrt{1-x^2}}\right) = 0$

$$\Rightarrow \quad 1 - x^2 - x^2 = 0$$
$$\Rightarrow \quad 2x^2 = 1$$
$$\Rightarrow \quad x^2 = \tfrac{1}{2}$$
$$\Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}} = \pm\tfrac{\sqrt{2}}{2}$$

when $x = \frac{\sqrt{2}}{2}$, $y = \sqrt{1-(\frac{\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$

$x = \frac{-\sqrt{2}}{2}$, $y = \sqrt{1-(\frac{-\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$

So two critical pts: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

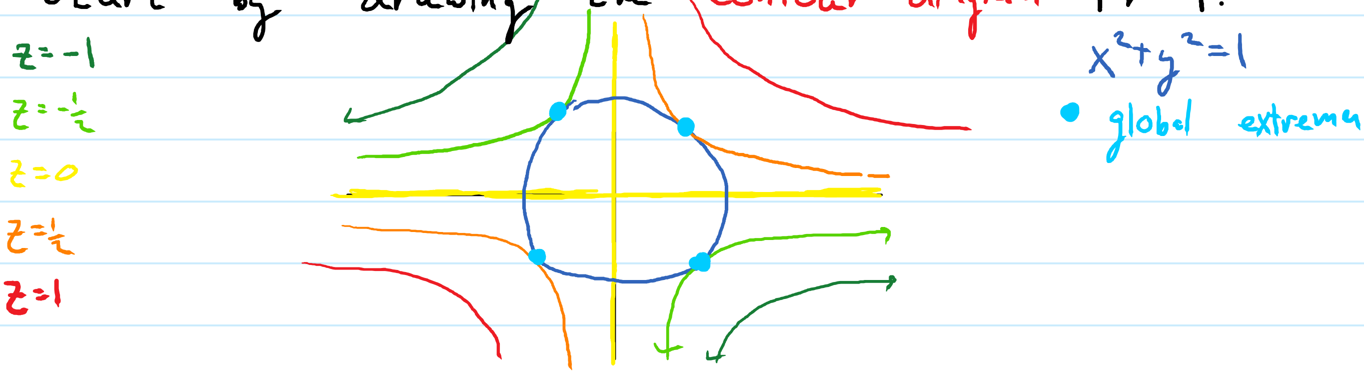
On S_2, we get critical pts at: $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

We must also include the end pts: $(1,0)$, $(-1,0)$.

plugging back into $f(x,y) = xy$, we see f has a global max of $\frac{1}{2}$ occurring at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and a global min. of $-\frac{1}{2}$ occurring at $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Let's think of a better way to do this:

Start by drawing the contour diagram for f.

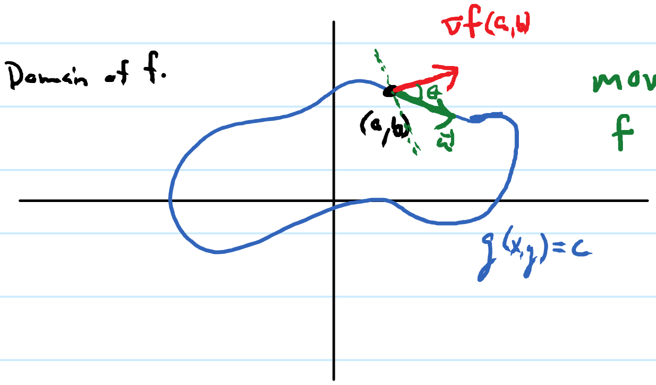


If we draw the constraint: $x^2+y^2=1$ on top of the contour diagram, we see that at the points where f achieves its global max./min., the constraint $x^2+y^2=1$ is tangent to the level curve of f.

In fact, this is how we find global extrema.

Why does this work? In the general setting, suppose f is cont., g is cont., and c is a constant. Suppose we want to find the max./min.s of $f(x,y)$ subject to the constraint $g(x,y)=c$. If (a,b) is a max. of f on $g(x,y)=c$, then we claim $\nabla f(a,b)$ be $\perp$ to the constraint curve.

<u>Proof in the special case where g, f are both diff'ble:</u> Assume $\nabla f(a,b)$ is not $\perp$ to $g(x,y)=c$. Then the angle θ between $\nabla f(a,b)$ and $g(x,y)=c$ is $< \frac{\pi}{2}$. Let $\vec{u}$ be a unit vector tangent to $g(x,y)=c$.



movement in this direction increases f value.

Then
$$
\begin{aligned}
[D_{\vec{u}} f](a,b) &= (\nabla f)(a,b) \cdot \vec{u} \\
&= \|\nabla f(a,b)\| \|\vec{u}\| \cos\theta \\
&= \|\nabla f(a,b)\| \cos\theta .
\end{aligned}
$$

Since $\theta < \frac{\pi}{2}$, $\cos\theta > 0 \Rightarrow [D_{\vec{u}} f](a,b) > 0$. This means if we walk along the constraint $g(x,y)=c$ in the $\vec{u}$ direction, the value of f increases. This contradicts the fact that (a,b)

was supposed to be a max of f on $g(x,y)=c$. $\boxed{\text{VV}}$.

Corollary: ① Since $\nabla f(a,b)$ is $\perp$ to the level curve of f at (a,b) and $\nabla f(a,b)$ is $\perp$ to $g(x,y)=c$, the level curve of f at (a,b) is $\parallel$ (and hence tangent) to $g(x,y)=c$.

② Since $(\nabla g)(a,b)$ and $(\nabla f)(a,b)$ are both $\perp$ to $g(x,y)=c$, $\boxed{\nabla g(a,b) \text{ is } \parallel \text{ to } (\nabla f)(a,b).}$

Part ② of the corollary gives us:

Theorem: (The method of Lagrange multipliers): Let f, g be "nice" fcns from $\mathbb{R}^n \to \mathbb{R}$ (f, g have continuous partials). If $\vec{X}_0$ is a global min. or max. of f subject to the constraint $g(\vec{x})=c$, and if $(\nabla g)(\vec{x}_0) \neq \vec{0}$, then $(\nabla g)(\vec{x}_0) \parallel (\nabla f)(\vec{x}_0)$.
I.e., there is a constant, λ, such that

$$(\nabla f)(\vec{x}_0) = \lambda (\nabla g)(\vec{x}_0)$$

Warning: ① This theorem does not guarantee the existence of extreme values. If $f|_{g=c}$ does have extreme values, this is how you find them.

② The points where $\nabla g = \vec{0}$ are considered critical points. This almost never comes up.

Example: Use the method of Lagrange multipliers to find extreme values of $f(x,y) = xy$ subject to the constraint $x^2+y^2=1$.

Sol: Since $x^2+y^2=1$ is closed and bounded, extreme values exist.

Let $g(x,y) = x^2+y^2$. Then the constraint is $g(x,y)=1$.

Note! $\nabla g(x,y) = (2x, 2y)$. Since $\nabla g = \vec{0}$ only at $(0,0)$, which is not on the unit circle $x^2+y^2=1$, we can move on to solving the Lagrange Condition:

$$\nabla f = \lambda \nabla g.$$

Since $\nabla f = (y,x)$, we must solve

$$(y,x) = \lambda(2x, 2y). \ \& \ x^2+y^2=1$$

$\Rightarrow$ I $y = \lambda 2x$

II $x = \lambda 2y$

III $x^2+y^2=1$

General strategy 1: Solve for λ in terms of x,y.

I $y = 2\lambda x \Rightarrow$ either $\lambda = \frac{y}{2x}$ or $x=0$, in which case, dividing by x doesn't make sense!

However, if $x=0$, we get $y = 2\lambda(0) = 0$, which implies $(x,y) = (0,0)$ which is not on the unit circle.

Therefore it is impossible for $x=0$, and we therefore know $\boxed{\lambda = \frac{y}{2x}}$

II $x = 2\lambda y \Rightarrow \lambda = \frac{x}{2y}$ or $y=0$, in which case dividing by 0 doesn't make sense. By the same reasoning as above, y cannot be 0. Therefore $\boxed{\lambda = \frac{x}{2y}}$

$$\Rightarrow \frac{y}{2x} = \frac{x}{2y} \Rightarrow 2y^2 = 2x^2 \Rightarrow \boxed{y^2 = x^2 \Rightarrow y = \pm x}$$

III $x^2+y^2=1 \Rightarrow x^2+x^2=1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow \boxed{x = \frac{1}{\sqrt{2}} \text{ or } \frac{-1}{\sqrt{2}}.}$

If $x = \frac{1}{\sqrt{2}}$, $y = \pm x = \pm\frac{1}{\sqrt{2}}$. So we get two critical points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

If $x = -\frac{1}{\sqrt{2}}$, $y = \pm x \Rightarrow y = \mp\frac{1}{\sqrt{2}}$, So we have two more critical pts: $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Finally!

$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{2}$$
$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right) = \tfrac{-1}{2}$$
$$f\left(\tfrac{-1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right) = \tfrac{1}{2}$$
$$f\left(\tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \tfrac{-1}{2}$$

global max of $\frac{1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

global min. of $\frac{-1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Next time we will do examples until we run out of time.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 27, 2017

Section:
- §3.4

Topics Covered:
- Examples using the method of Lagrange multipliers

From last time:

Let f, g be "nice" fcns (f, g have continuous partials). Suppose $\vec{x}_0$ is a global extreme of f subject to the constraint $g = c$. If $(\nabla g)(\vec{x}_0) \neq \vec{0}$, then

$$\boxed{(\nabla f)(\vec{x}_0) = \lambda (\nabla g)(\vec{x}_0)}$$

Examples: ① Find the max and min of $f(x,y,z) = x - y + 2z$ on the closed and bounded domain $\{(x,y,z) \mid x^2+y^2+z^2 = 2\}$

Note: the domain is the set of all point on the sphere of radius $\sqrt{2}$. In particular, the domain is closed and bounded. By the EVT, f does have a max and min. on $x^2+y^2+z^2=2$.

Let $g(x,y,z) = x^2+y^2+z^2$. Note that $\nabla g = (2x, 2y, 2z) = \vec{0}$ only at the origin $(0,0,0)$, which is not on the sphere. So we can disregard this point and move on to solving the Lagrange equations.

$$\nabla f = \lambda g$$

Since $\nabla f = (1, -1, 2)$, we solve $(1,-1,2) = \lambda(2x, 2y, 2z)$. We have:

$$\text{I} \quad 1 = 2\lambda x$$
$$\text{II} \quad -1 = 2\lambda y$$
$$\text{III} \quad 2 = 2\lambda z$$
$$\text{IV} \quad 2 = x^2+y^2+z^2.$$

General strategy #2: Solve for x, y, z in terms of λ.

Note that λ cannot be 0, since otherwise eqn (I) turns into $1=0$. Therefore it is safe to divide by λ.

$$\text{(I)} \quad 1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda}$$
$$\text{(II)} \quad -1 = 2\lambda y \Rightarrow y = \frac{-1}{2\lambda}$$
$$\text{(III)} \quad 2 = 2\lambda z \Rightarrow z = \frac{1}{\lambda}$$

Now (IV) gives $x^2+y^2+z^2=2 \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 2$

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 2$$
$$\Rightarrow \frac{3}{2\lambda^2} = 2$$
$$\Rightarrow \lambda^2 = \frac{3}{4}$$
$$\Rightarrow \boxed{\lambda = \pm\frac{\sqrt{3}}{2}}$$

If $\lambda = \frac{\sqrt{3}}{2}$,
$$x = \frac{1}{2\lambda} = \frac{1}{\sqrt{3}}, \quad y = \frac{-1}{2\lambda} = \frac{-1}{\sqrt{3}}, \quad z = \frac{1}{\lambda} = \frac{2}{\sqrt{3}}$$

If $\lambda = -\frac{\sqrt{3}}{2}$,
$$x = \frac{-1}{\sqrt{3}}, \quad y = \frac{1}{\sqrt{3}}, \quad z = \frac{-2}{\sqrt{3}}$$

There are two critical points: $\boxed{\frac{1}{\sqrt{3}}(1,-1,2), \ \frac{1}{\sqrt{3}}(-1,1,-2)}$

$f\left(\frac{1}{\sqrt{3}}(1,-1,2)\right) = \frac{4}{\sqrt{3}}$ global max.

$f\left(\frac{1}{\sqrt{3}}(-1,1,-2)\right) = -\frac{4}{\sqrt{3}}$ global min.

<u>Example 2:</u> Find the point on the ellipse $x^2+y^2=1$ that is closest to the point $(4,-2)$.

<u>Sol:</u> We first must find a fcn to optimize.

Recall that if (x,y) is a point on the plane, the distance from (x,y) to $(4,-2)$ is given by the fcn $d_0(x,y) = \sqrt{(x-4)^2+(y+2)^2}$. We want the point on the constraint $g(x,y) = x^2+y^2=1$ that minimizes d.

Since d_0 is minimal exactly where $d(x,y) = (x-4)^2+(y+2)^2$ we will minimize $d(x,y)$ instead.

Note: ① $x^2+y^2=1$ is closed and bounded, and d is cont. $\Rightarrow$ EVT tells us extrema exist.

② $\nabla g = \vec{0}$ only at the point $(0,0)$ which is not on the ellipse, so we ignore.

We solve: $\nabla f = \lambda \nabla g$

$$\Rightarrow \big(2(x-4),\, 2(y+2)\big) = \lambda(2x, 2y) \quad \text{and} \quad x^2+y^2=1.$$

$$\Rightarrow \begin{array}{ll} \text{I} & 2(x-4) = 2\lambda x \\ \text{II} & 2(y+2) = 2\lambda y \\ \text{III} & x^2+y^2=1 \end{array}$$

I: Note that if x were 0, I says $2(-4)=0$, which is nonsense. Therefore, we are safe to divide by x since $x \neq 0$. I $\Rightarrow$ $\boxed{\lambda = \frac{x-4}{x}}$.

II: If y were 0, II becomes $(2)(2)=0$, which is nonsense. Therefore, $y \neq 0$ and we are free to divide by y.

II $\Rightarrow$ $\boxed{\lambda = \frac{y+2}{y}}$

As $\lambda = \frac{x-4}{x}$ and $\frac{y+2}{y}$, we have: $\frac{x-4}{x} = \frac{y+2}{y}$ $\Rightarrow$

$$xy - 4y = xy + 2x \Rightarrow \boxed{x = -2y}$$

Now III says $x^2 + y^2 = 1 \Rightarrow (-2y)^2 + y^2 = 1$

$\Rightarrow y = \pm \frac{1}{\sqrt{5}}$

So we get two points: $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$.

It is easy to check $\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ is the closest to $(4,-2)$ on the unit circle.

__Rmk:__ This answer is obvious because $\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ is the normalization of $(4,-2)$.

__Example 3:__ Find the maximum of $f(x,y) = e^{-x^2-y^2}$ the constraint $3y + 2x = 4$.

__Note:__ The constraint $3y + 2x = 4$ is a line, and is therefore __not__ __bounded__. We are not guaranteed to have extrema!

not bounded.

Let's first try to find critical points, and then prove that they do in fact give us the max.

Since $\nabla g = (2,3) \neq \vec{0}$, we move on to solving the Lagrange eqn:

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \quad \left(-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}\right) = \lambda(2,3) \quad \text{and} \quad 3y+2x=4.$$

$$\Rightarrow \text{I} \quad -2xe^{-x^2-y^2} = 2\lambda$$
$$\text{II} \quad -2ye^{-x^2-y^2} = 3\lambda$$
$$\text{III} \quad 3y+2x=4$$

$$\left.\begin{array}{l} \text{I} \Rightarrow \quad \lambda = -xe^{-x^2-y^2} \\ \text{II} \Rightarrow \quad \lambda = -\frac{2}{3}ye^{-x^2-y^2} \end{array}\right\} \Rightarrow xe^{-x^2-y^2} = \frac{2}{3}ye^{-x^2-y^2}$$

As $e^{-x^2-y^2} \neq 0 \Rightarrow \boxed{x = \frac{3}{2}y}$

$$\text{III} \Rightarrow \quad 3y + 2\left(\frac{3}{2}y\right) = 4$$
$$\Rightarrow y = \frac{2}{3} \Rightarrow x = 1$$

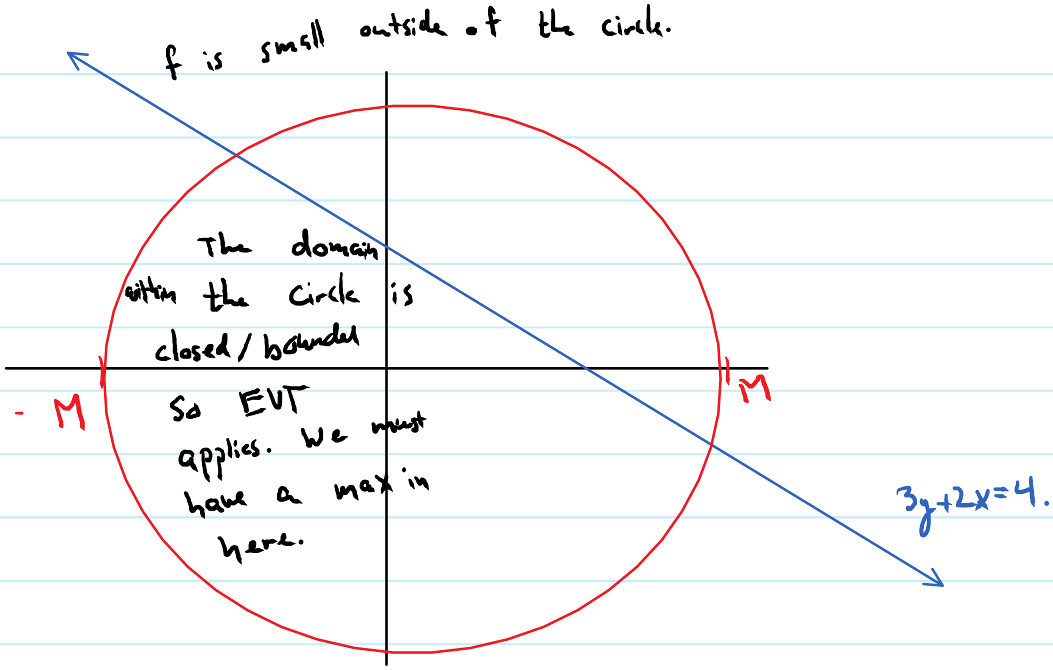
we have one critical point: $\boxed{(1, \frac{2}{3})}$

Now we prove f has a max. at $(1, \frac{2}{3})$.

First note $f(1, \frac{2}{3}) = e^{-1-\frac{4}{9}} = \boxed{e^{-\frac{13}{9}}}$

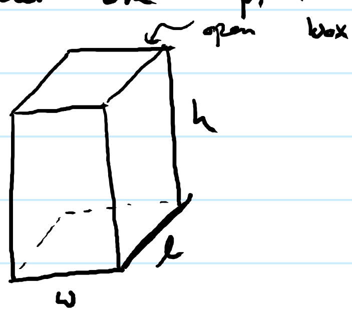
Second, note $\displaystyle\lim_{\|(x,y)\| \to \infty} \left(e^{-x^2-y^2}\right) = 0$. I.e., as (x,y) goes to infinity in any direction, $f(x,y) \to 0$. So there is a constant M, so that if $\|(x,y)\| > M$, $f(x,y) < \frac{1}{2}e^{-13/9}$.

Therefore, f restricted to $3y+2x=4$ attains its max within the closed and bounded set $\{(x,y) \mid 3y+2x=4 \ \& \ \|(x,y)\| \leq M\}$. Since $(1, \frac{2}{3})$ was the only point we found, it must actually be the max.



Example 4: What is the maximal area of an open top box with surface area 64 cm^2?

Sol: Consider the picture below:



We want to minimize the fcn $V(w,l,h) = wlh$ given that Surface area $= wl + 2lh + 2wh = 64$.

Let $g(w,l,h) = wl + 2lh + 2wh$.

The constraint: $wl + 2lh + 2wh = 64$ is unbdd, so it is unclear if a max exists. We show later that we may assume $0 \le w,l,h \le 64$. (See last page).

$\nabla V = (lh,\ wh,\ wl)$
$\nabla g = (l+2h,\ w+2h,\ 2l+2w).$

$\nabla g = \vec{0}$ only at the origin which is not on our constraint.

So we solve: $\nabla f = \lambda \nabla g$

$$\Rightarrow (lh,\ wh,\ wl) = \lambda(l+2h,\ w+2h,\ 2l+2w)$$

$$\Rightarrow \begin{array}{ll} \text{I} & lh = \lambda(l+2h) \\ \text{II} & wh = \lambda(w+2h) \\ \text{III} & lw = \lambda(2l+2w) \\ \text{IV} & wl + 2wh + 2lh = 64 \end{array}$$

We may assume $w, l, h \neq 0$ since that would give $V(w,l,h) = 0$.

$$\left.\begin{array}{l} \text{I} \Rightarrow \lambda = \frac{lh}{l+2h} \\ \text{II} \Rightarrow \lambda = \frac{wh}{w+2h} \end{array}\right\} \Rightarrow \frac{lh}{l+2h} = \frac{wh}{w+2h}$$

$$\text{III} \Rightarrow \lambda = \frac{lw}{2l+2w}$$

$$\Rightarrow l(w+2h) = w(l+2h)$$
$$\Rightarrow \cancel{lw} + 2lh = \cancel{lw} + 2wh$$
$$\Rightarrow 2lh = 2wh$$
$$\Rightarrow \boxed{l = w}$$

Now: $\frac{wh}{w+2h} = \frac{lw}{2l+2w} \Rightarrow \frac{wh}{w+2h} = \frac{w^2}{4w} = \frac{w}{4}$

$$\Rightarrow 4wh = w^2 + 2wh$$
$$\Rightarrow 0 = w^2 - 2wh$$
$$\Rightarrow 0 = w(w-2h)$$
$$\Rightarrow \cancel{w=0} \quad \text{or} \quad w = 2h$$
$$\Rightarrow \boxed{l = w = 2h}$$

gives $Vol = 0$

Finally, IV: $wl + 2lh + 2wh = 64$

$\Rightarrow \quad 4h^2 + 4h^2 + 4h^2 = 64$

$\Rightarrow \quad 12h^2 = 64$

$h^2 = \frac{16}{3}$

$\Rightarrow \quad \boxed{h = \frac{4}{\sqrt{3}}}$ (h can't be ≤0!)

$\Rightarrow \quad \boxed{w = \frac{8}{\sqrt{3}}, \quad l = \frac{8}{\sqrt{3}}}$

$\Rightarrow$ max. volume $= \left(\frac{8}{\sqrt{3}}\right)\left(\frac{8}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right) = \boxed{\frac{256}{3\cdot\sqrt{3}}}$

If say $w > 64$,

then we get $64 = wl + 2lh + 2wh \geq wl$

$$\Rightarrow \quad 64 \geq wl$$

Similarly $64 = wl + 2lh + 2wh \geq 2wh$

$$32 > wh$$

$$64 > wl \quad 32 > wh.$$

$$\Rightarrow \quad w^2 l h < 64 \cdot 32 \Rightarrow \text{vol} \leq \frac{64 \cdot 32}{w} < 32$$

Similarly, if $l > 64$, $\text{Vol} \leq 32$,

if $h > 64$, $64 = wl + 2wh + 2lh > 2lh$

$$\Rightarrow \quad 32 > lh$$

similarly, $64 = wl + 2wh + 2lh > 2wh$

$$\Rightarrow \quad 32 > wh$$

$$\Rightarrow \quad 32^2 > wlh^2$$

$$\Rightarrow \quad \frac{32^2}{h} > wlh = \text{Vol}$$

but $\text{vol} < \frac{32^2}{h} < \frac{32^2}{64} < 16.$

So outside of the bounded region, $0 \leq w, l, h \leq 64$, $\text{Vol} \leq 32$.

We see that inside the bounded region $0 \leq w, l, h \leq 64$, we can find a point with larger volume than 32. Therefore that point must actually be a global max.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 6, 2017

Section:
- [ ] § 4.1

Topics Covered:
- [ ] Finding position from acceleration
- [ ] Newton's 2nd Law of Motion

§4.1: More on paths and Newton's 2nd Law of Motion:

Recall: If $\vec{c}:\mathbb{R}^n \to \mathbb{R}$ $(n=2 \text{ or } 3)$ is a path fcn

$$\vec{c}(t) = (x(t), y(t)) \text{ or } (x(t), y(t), z(t)),$$

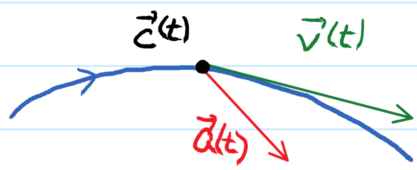
we think of $\vec{c}(t)$ as the position of a particle floating in space at time t. As t progresses, we wish to track the motion of the particle.

To understand the trajectory of the particle, we consider it's velocity vector $\vec{v}(t) = \vec{c}\,'(t) = (x'(t), y'(t))$ or $(x'(t), y'(t), z'(t))$.
we get the speed of the particle by taking the magnitude of velocity:

$$s(t) = \|\vec{c}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2} \quad \left(= \sqrt{\vec{c}\,'(t)\cdot\vec{c}\,'(t)}\right)$$

Finally, we track the change in velocity by looking at the acceleration vector

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$$



Review: Special Case of differentiation rules:

Let $\vec{c}(t)$, $\vec{r}(t)$ be path fcns. Then

① $\frac{d}{dt}\left((\vec{c}+\vec{r})(t)\right) = \vec{c}\,'(t) + \vec{r}\,'(t)$

② $\frac{d}{dt}\left((\alpha\vec{c})(t)\right) = \alpha\,\vec{c}\,'(t)$

③ $\frac{d}{dt}\left(\vec{c}(t)\cdot\vec{r}(t)\right) = \vec{c}\,'(t)\cdot r(t) + \vec{c}(t)\cdot\vec{r}\,'(t)$

④ $\frac{d}{dt}\left(\vec{c}(t)\times\vec{r}(t)\right) = \vec{c}\,'(t)\times\vec{r}(t) + \vec{c}(t)\times\vec{r}\,'(t)$.

Ex: Use the product rule to find $\frac{d}{dt}(\vec{c}(t)\cdot\vec{r}(t))$ where

$$\vec{c}(t) = (t, 1)$$
$$\vec{r}(t) = (\cos(t), \sin(t))$$

$$\frac{d}{dt}(\vec{c}(t)\cdot\vec{r}(t)) = \vec{c}\,'(t)\cdot\vec{r}(t) + \vec{c}(t)\cdot\vec{r}\,'(t)$$
$$= (1,0)\cdot(\cos(t), \sin(t)) + (t,1)\cdot(-\sin(t), \cos(t))$$
$$= \cos(t) - t\sin(t) + \cos(t)$$
$$= \boxed{2\cos(t) - t\sin(t)}$$

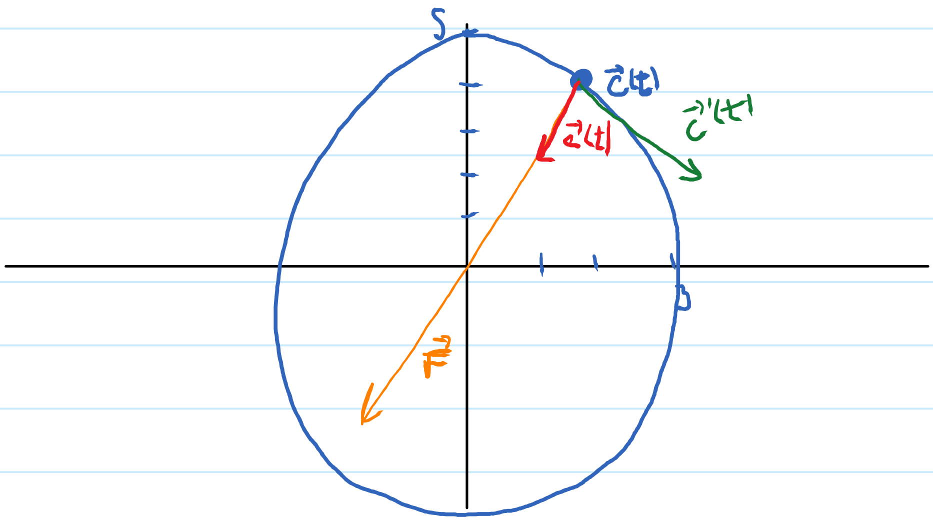
Application: Newton's 2nd Law of Motion:

Force = (mass) x (acceleration)

$$\boxed{\vec{F} = m\vec{a}}$$

Example: A 150 kg mass object travels an elliptical orbit $\vec{c}(t) = (3\sin(t), 5\cos(t))$

(Here distance is meters, time is in sec.)



Calculate the force acting on the object.

Sol: $\vec{c}\,'(t) = (3\cos(t), -5\sin(t))$

$\vec{a}''(t) = (-3\sin(t), 5\cos(t))$

Therefore, the force vector $\vec{F} = m\vec{a}$

$= 150(-3\sin(t), 5\cos(t))$

Equations of motion (vector version):-

In calc. 2, you learned that if you know:

- acceleration
- initial velocity
- initial position

of an object (in a vacuum) then you can fully describe the motion of the object. We can do the same with vector valued fcns. For example, since $\vec{c}\,'(t) = \vec{v}(t)$ for some path fcn $\vec{c}(t)$, we can integrate the components of $\vec{v}(t)$ to recover $\vec{c}(t)$ (up to a constant). If we also know some initial value of $\vec{c}$, you can determine $\vec{c}$ uniquely.

Example: Let $\vec{c}(t)$ be a path fcn. Suppose

- $\vec{v}(t) = (t, t^2, 0)$
- $\vec{c}(1) = (0, 3, 1)$

① Find a formula for $\vec{c}(t)$.

② Evaluate $\vec{c}(5)$.

③ Find the displacement vector, $\vec{c}(5) - \vec{c}(1)$, time $t=1$ to $t=5$.

Sol: ① Since $\vec{c}\,'(t) = \vec{v}(t) = (t, t^2, 0)$

we must have $\vec{c}(t) = \left(\int t\,dt, \int t^2 dt, \int 0\,dt\right)$

$$= \left(\tfrac{1}{2}t^2 + C_1,\ \tfrac{1}{3}t^3 + C_2,\ C_3\right)$$

for some constants C_1, C_2, C_3. We can write this more succinctly as $\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \vec{u}$ where $u = (C_1, C_2, C_3)$ is a constant vector.

to find $\vec{u}$ we use the fact that $\vec{c}(1) = (0,3,1)$. I.e, if we plug in $t=1$, we get $(0,3,1)$.

$$\therefore \quad (0,3,1) = \vec{c}(1) = \left(\tfrac{1}{2}(1)^2, \tfrac{1}{3}(1)^3, 0\right) + \vec{u}$$
$$\Rightarrow \quad (0,3,1) = \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) + \vec{u}$$
$$\Rightarrow \quad \vec{u} = (0,3,1) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) = \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)$$

So $\boxed{\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)}$

Rmk: Alternatively, we could use the 2^{nd} fundamental Thm of calculus:

$$\vec{c}(t) - \vec{c}(1) = \int_1^t \vec{c}\,'(x)\,dx$$
$$\vec{c}(t) - \vec{c}(1) = \int_1^t (x, x^2, 0)\,dx$$
$$\vec{c}(t) - \vec{c}(1) = \left(\tfrac{1}{2}x^2, \tfrac{1}{3}x^3, 0\right)\Big|_{x=1}^{x=t}$$
$$\vec{c}(t) - (0,3,1) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right)$$
$$\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) + (0,3,1)$$
$$\boxed{= \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)}$$

(2) $\vec{c}(3) = \left(\tfrac{9}{2}, 9, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)$
$= \left(4, \tfrac{35}{3}, 1\right)$

(3) $\vec{c}(3) - \vec{c}(1) = \left(4, \tfrac{35}{3}, 1\right) - (0,3,1) = (4, 13, 0)$

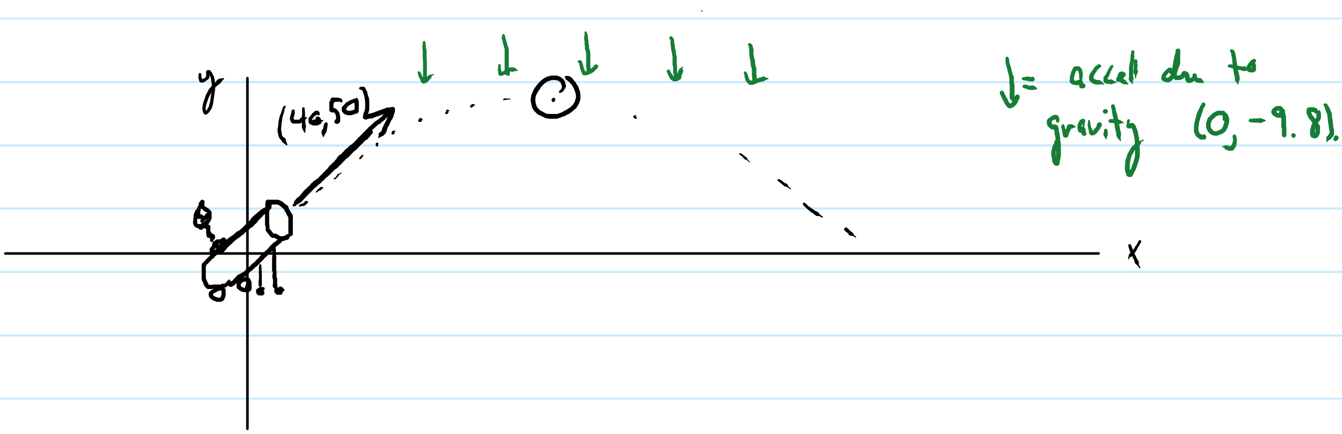
So the particle went 4 units in the positive x-dir.
13 units in positive y-dir.
0 units in z-dir.

Remark: if we only wanted part (c), we could do

$$\vec{c}(3) - \vec{c}(1) = \int_1^3 \vec{c}\,'(t)\,dt$$
$$\vec{c}(3) - \vec{c}(1) = \int_1^3 (t, t^2, 0)\,dt$$
$$= \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right)\Big|_{t=1}^{t=3} = \left(\tfrac{1}{2}(3)^2, \tfrac{1}{3}(3)^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right)$$
$$= \left(\tfrac{9}{2}, \tfrac{27}{3}, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) = (4, 13, 0)$$

at the point (0,0)

Ex: A cannonball is shot out of a canon. At t=0 seconds, it's velocity is given by (50,50) (Here the units are m/s). Find the fcn that describes the motion of the ball.



here, $\vec{a}(t) = (0, -9.8)$ (accel. due to gravity)

$$\vec{v}(t) = \left(\int 0\,dt, \int -9.8\,dt\right)$$
$$= (0, -9.8t) + \vec{v}_0$$ where $\vec{v}_0$ is a constant vector.

Since $\vec{v}(0) = (40,50)$, we have
$$(40,50) = (0,0) + \vec{v}_0 \Rightarrow \vec{v}_0 = (40,50).$$

$\Rightarrow \vec{v}(t) = (40, -9.8t + 50)$. If $\vec{c}(t)$ is the position vector,
$\vec{c}(t) = (40t, -4.9t^2 + 50t) + \vec{c}_0$. where $\vec{c}_0$ is a constant vector.

Since $\vec{c}(0) = (0,0)$, we have

$$(0,0) = \vec{c}(0) = (0,0) + \vec{c}_0 \Rightarrow \vec{c}_0 = (0,0)$$

$$\Rightarrow \boxed{\vec{c}(t) = (40t, -4.9t^2 + 50t).}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 8, 2017

Section:

- §4.2

Topics Covered:

- Arc Length

§4.2: Arc Length Formula:

Q? If you drive in a straight line going 60 mph for 4 hours, how far did you travel?

Sol. I'll give you a moment to think,

...,

.. ,

...

Okay, so the total distance travelled is

speed × time $(60\ \text{m/h})(4\text{h}) = 240\text{m}$.

More generally, you probably learned in Calc. 2, that if you integrate speed over some time interval, you get the total distance travelled over that time interval.

So our intuition tells us:

Thm. Let $\vec{c}(t)$ be a diff'ble path fcn defined on the (time) interval $a \leq t \leq b$, then the (arclength / total distance travelled) is given by

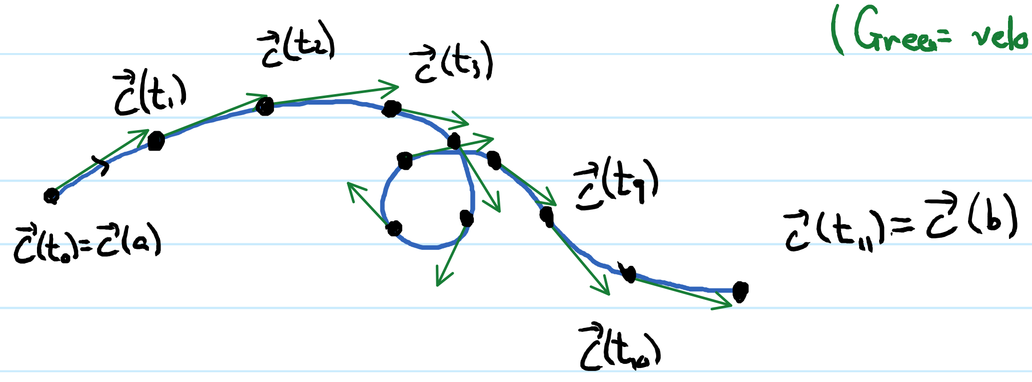
$$\int_a^b (\text{speed})\,dt = \int_a^b \|\vec{c}\,'(t)\|\,dt$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \quad (\text{in } \mathbb{R}^2)$$

$$\left(\text{or} \quad = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \quad (\text{in } \mathbb{R}^3)\right)$$

Informal "proof": Let $\vec{c}(t)$ be a diff'ble path fcn defined on the time interval $a \le t \le b$.

Idea: Split $[a,b]$ into N many small pieces, of equal length and for $i=0,1,\dots,N$, let t_i be the endpt. of the i^{th} subinterval:



the arclength traced by $\vec{c}$ on the time subinterval $[t_i, t_{i+1}]$ can be approximated by:

$$(\text{speed at time } t_i) \times (\text{time elapsed}) \qquad \left(\text{See green text for details}\right)$$

$$= s(t_i) \times (t_{i-1} - t_i)$$

$$= s(t_i)\Delta t$$

where $s(t_i) = \|\vec{c}\,'(t_i)\|$, $\Delta t = t_{i-1} - t_i$.

So
$$\text{arclength} = \sum_{i=1}^{N} (\text{arclength on the } i^{th} \text{ subinterval})$$
$$\approx \sum_{i=1}^{N} \|\vec{c}\,'(t_i)\|\,\Delta t.$$

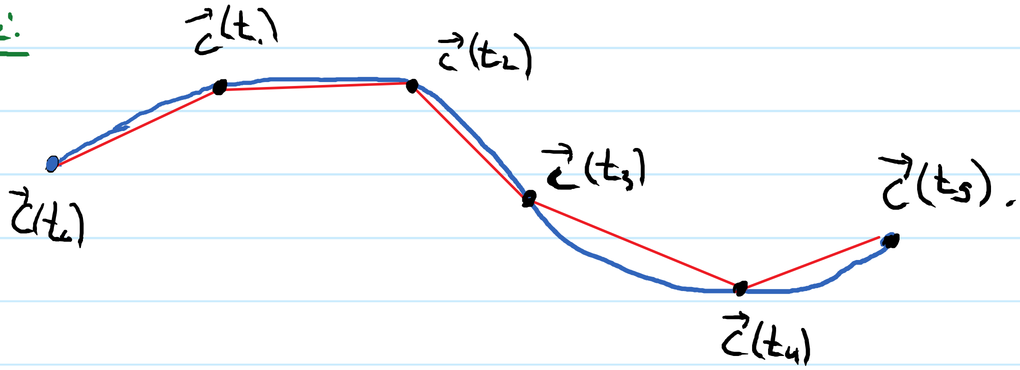
As the number of subintervals, $N \to \infty$, the estimate is more and more accurate. Therefore,

$$\text{arclength} = \lim_{N\to\infty}\left(\sum_{i=1}^{N} \|\vec{c}\,'(t)\|\Delta t\right)$$

This is a limit of Riemann Sums, which is the definition of the (definite) integral.

$$\Rightarrow \quad \text{arclength} = \lim_{N\to\infty}\left(\sum_{i=1}^{N} \|\vec{c}\,'(t)\|\,\Delta t\right)$$
$$= \int_a^b \|\vec{c}\,'(t)\|\,dt.$$

Picture:



Approximate arclength on each subinterval by a straight line. For the i^{th} subinterval, the length of the line segment is

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \left(\frac{\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\|}{t_{i+1} - t_i}\right)(t_{i+1} - t_i).$$

If $\Delta t := t_{i+1} - t_i$ is small (i.e., # of subintervals is large), then

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \left(\frac{\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\|}{\Delta t}\right)\Delta t$$

$$= \|\vec{c}\,'(t_*)\|\,\Delta t$$

where t_* is a # s.t. $t_i \le t_* \le t_{i+1}$ (here we used mean value thm).

Example: (1) Find the length of the path traced by the fcn $\vec{c}(t) = (4\cos(2t), 4\sin(2t))$ from $t=0$ to $t=2\pi$.

Sol: Rmk: $\vec{c}(t)$, $0 \le t \le 2\pi$ travels the circumference of a circle of radius 4 twice. We should expect arclength to be $2 \times (\text{circumference}) = (2)(8\pi) = 16\pi$.

$$\vec{c}\,'(t) = (-8\sin(2t), 8\cos(t)).$$

$$\|\vec{c}\,'(t)\| = \left((-8\sin(2t))^2 + (8\cos(2t))^2\right)^{1/2}$$
$$= (64\sin^2(2t) + 64\cos^2(2t))^{1/2}$$
$$= (64)^{1/2}$$
$$= 8.$$

$$\text{Arclength} = \int_0^{2\pi} 8\,dt = 8t\Big|_0^{2\pi}$$
$$= 16\pi - 0$$
$$= \boxed{16\pi}$$

(2) Find the length of the path travelled by

$$\vec{c}(t) = \left(\ln(\sqrt{t}), -\sqrt{3}\,t, \tfrac{3}{2}t^2\right) \quad \text{from } t=1 \text{ to } t=2.$$

Sol: $$\vec{c}\,'(t) = \left(\left(\frac{1}{\sqrt{t}}\right)\left(\frac{1}{2\sqrt{t}}\right), \sqrt{3}, 3t\right)$$
$$= \left(\frac{1}{2t}, \sqrt{3}, 3t\right).$$

$$\|\vec{c}\,'(t)\| = \left(\left(\frac{1}{2t}\right)^2 + (\sqrt{3})^2 + (3t)^2\right)^{1/2}$$
$$= \left(\frac{1}{4t^2} + 3 + 9t^2\right)^{1/2}$$
$$= \left(\frac{1 + 12t^2 + 36t^4}{4t^2}\right)^{1/2}$$

$$= \left(\frac{(6t^2+1)^2}{4t^2} \right)^{1/2}$$

$$= \frac{6t^2+1}{2t} = \frac{6t^2}{2t} + \frac{1}{2t}$$

$$= \boxed{3t + \frac{1}{2t}}$$

$$\Rightarrow \quad \text{arclength} = \int_1^2 \|\vec{c}\,'(t)\| \, dt$$

$$= \int_1^2 3t + \frac{1}{2t} \, dt$$

$$= \left(\frac{3}{2}t^2 + \frac{1}{2}\ln|t| \right) \Big|_1^2$$

$$= \left(\frac{3}{2}(2)^2 + \frac{1}{2}\ln(2) \right) - \left(\frac{3}{2} + \frac{1}{2}\cancelto{0}{\ln(1)} \right)$$

$$= \left(6 + \frac{1}{2}\ln(2) - \frac{3}{2} \right)$$

$$= \boxed{\frac{9}{2} + \frac{1}{2}\ln(2)}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 9, 2017

Section:

- [ ] § 5.1

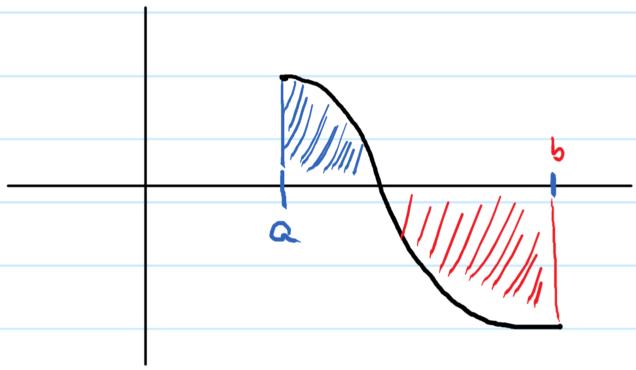
Topics Covered:

- [ ] Volume and Cavalieri's Principle
- [ ] Double integration as iterated integrals
- [ ] Fubini's Theorem

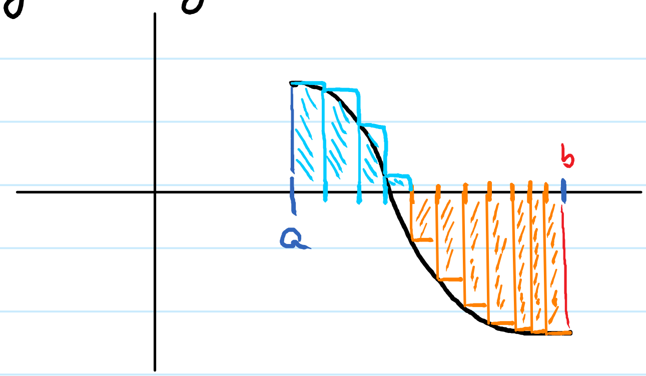
§5.1: Double Integrals as Volume & Cavalieri's Principle:

If $f(x)$ is a continuous fcn on the closed interval $[a,b]$, you learned that the (signed) area between the graph of f and the x-axis is given by the definite integral

$$\int_a^b f(x)\,dx = (\text{blue area}) - (\text{red area})$$



Recall that in order to define the integral, we slice up the interval $[a,b]$ into N many equal parts, and approximate the area by rectangles

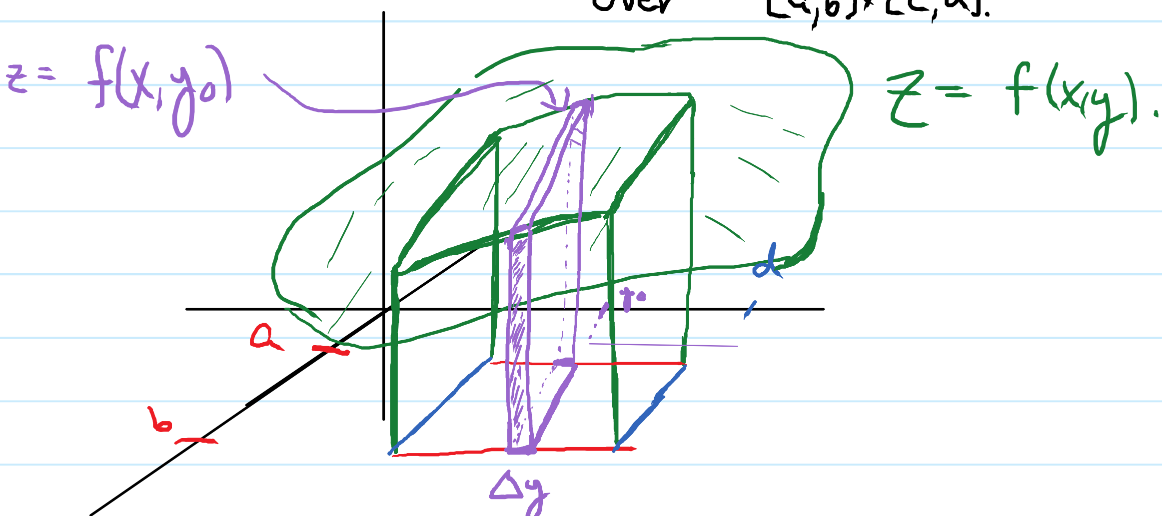


Then the integral is roughly the sum of the areas of the rectangles. As $N \to \infty$, the approximation $\underbrace{\sum_{i=1}^{N} f(x_i)\Delta x}_{\text{Right hand Riemann Sum}} \to \int_a^b f(x)\,dx$

This process is an example of **Cavalieri's Principle**: In order to find area/volume of a region, we slice it into pieces, find the area/volume of each slice and add them all up. As the number of slices $\to \infty$, we get the area/volume exactly.

Let $f(x,y)$ be a continuous fcn defined on a rectangle $[a,b]\times[c,d] = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$.
($[a,b]\times[c,d]$ is called the Cartesian Product of $[a,b]$ and $[c,d]$.)

We wish to find the (signed) volume of the region between the graph of f and the xy-plane over $[a,b]\times[c,d]$.



If we fix the y-variable ($y=y_0$) and slice the volume into pieces that are $\parallel$ to the xz-plane, we can approximate the value of the slice by

$$(\Delta y)\times(A(y))$$

where $A(y)$ is the area of the face of the slice. If M is the number of slices, then

$$\mathrm{Vol} = \underbrace{\lim_{M\to\infty}\left(\sum_{i=1}^{M} A(y)\Delta y\right)}_{\text{definition of definite integral}} = \int_c^d A(y)\,dy$$

On the other hand, for any fixed y_0,

$A(y_0)$ is the area under the graph of the fcn $f(x,y_0)$. Therefore,

$$A(y_0) = \int_a^b f(x,y_0)\,dx$$

(integral w.r.t. x. y is considered a constant).

All together, Volume under $z=f(x,y)$ =

$$\int_c^d A(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \overset{\text{notation}}{=} \iint_{[a,b]\times[c,d]} f(x,y)\,dA$$

This is called the double integral.

In plain english, to get the volume, integrate $f(x,y)$ w.r.t. x from $x=a$ to $x=b$ (pretend y is constant). Then take the result, and integrate w.r.t. y from $y=c$ to $y=d$.

Ex. Find the volume under the paraboloid $z=x^2+y^2$ over the rectangle $[0,2]\times[-1,1]$.

Sol: Here, $0 \le x \le 2$, $-1 \le y \le 1$.

So

$$\text{Vol.} = \int_{-1}^{1}\int_0^2 x^2+y^2\,dx\,dy$$

$$= \int_{-1}^{1} \left(\tfrac{1}{3}x^3 + y^2 x \right)\Big|_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} \left(\tfrac{8}{3}+2y^2\right) - \left(0 + 0\cdot y^2\right) dy = \int_{-1}^{1} \tfrac{8}{3} + 2y^2\,dy$$

$$= \left(\frac{8}{3}y + \frac{2}{3}y^3\right)\Big|_{y=-1}^{y=1}$$

$$= \left(\frac{8}{3} + \frac{2}{3}(1)^3\right) - \left(-\frac{8}{3} + \frac{2}{3}(-1)^3\right)$$

$$= \frac{10}{3} - \left(-\frac{10}{3}\right)$$

$$= \boxed{\frac{20}{3}}$$

__Rmk:__ In the initial discussion, we could have fixed x first, and made slices $\parallel$ to the yz-plane. In the end, we would still get the same answer. This gives us:

__Thm: (Fubini):__ If f is cont. on $[a,b]\times[c,d]$

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

In other words, we can switch the order of integration.

__Ex:__ Compute $\displaystyle\int_0^1\int_0^2 \frac{x}{1+xy}\,dx\,dy$.

As written, we would need to compute the "inside" integral
(y is "constant")

$\displaystyle\int_0^2 \frac{x}{1+xy}\,dx$. We could try to substitute:

$$u = 1+xy$$

$$\Rightarrow du = y\,dx \quad\text{and}\quad x = \frac{u-1}{y}$$

$$\Rightarrow \int \frac{x}{1+xy}\,dx = \int \left(\frac{u-1}{y}\right)\left(\frac{1}{u}\right)\left(\frac{1}{y}\right)du.$$

This is one option... but it doesn't look fun instead, let's ignore our problems for now and change the order of integration. Pay attn to bounds!

$$\int_0^1\int_0^2 \frac{x}{1+xy}\,dx\,dy = \int_0^2\int_0^1 \frac{x}{1+xy}\,dy\,dx$$

Now the "inside integral" is

$$\int_0^1 \frac{x}{1+xy}\,dy \qquad (x \text{ is "constant"})$$

Let $u = 1+xy \Rightarrow \frac{du}{dy} = x$
$\Rightarrow du = \underline{x\,dy}$

This shows up in the integral! OH BABY!

Then
$$\int_0^1 \frac{x}{1+xy}\,dy = \int_{u(0)}^{u(1)} \frac{1}{u}\,du = \ln|u|\Big|_{u(0)}^{u(1)}$$
$$= \ln|1+xy|\Big|_{y=0}^{y=1}$$
$$= \ln|1+x| - \ln(1)$$
$$= \ln|1+x|.$$

Now
$$\int_0^2\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^2 \ln(1+x)\,dx$$
$$\overset{*}{=} (1+x)\ln(1+x) - (1+x)\Big|_0^2$$
$$= (3\ln(3) - 3) - ((1)\ln(1) - 1)$$
$$= 3\ln(3) - 3 + 1$$
$$= \boxed{3\ln(3) - 2}$$

(*)

sub $\boxed{\begin{array}{l} w = x+1 \\ dw = dx \end{array}}$

$$\int \ln(x+1)\,dx = \int \ln(w)\,dw$$

Integration by parts

$$\boxed{\begin{array}{ll} u = \ln w & dv = dw \\ du = \frac{1}{w}\,dw & v = w \end{array}}$$

$$\int \ln(w)\,dw = w\ln(w) - \int dw = w\ln(w) - w$$
$$= \underline{(x+1)\ln(x+1)} - \underline{(x+1)}$$

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 13, 2017

Section:

- [ ] §5.2
- [ ] §5.3 (intro)

Topics Covered:

- [ ] The definition of the double integral
- [ ] Fubini's Theorem
- [ ] Introduction to integrals over more general regions

§ 5.2 Definition of the double integral using Riemann Sums:

Fix a rectangle $R = [a,b] \times [c,d]$ in $\mathbb{R}^2$. Let $f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuous. We saw last time that

$$\iint_R f\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy \underset{\text{Fubini}}{=} \int_a^b \int_c^d f(x,y)\,dy\,dx$$

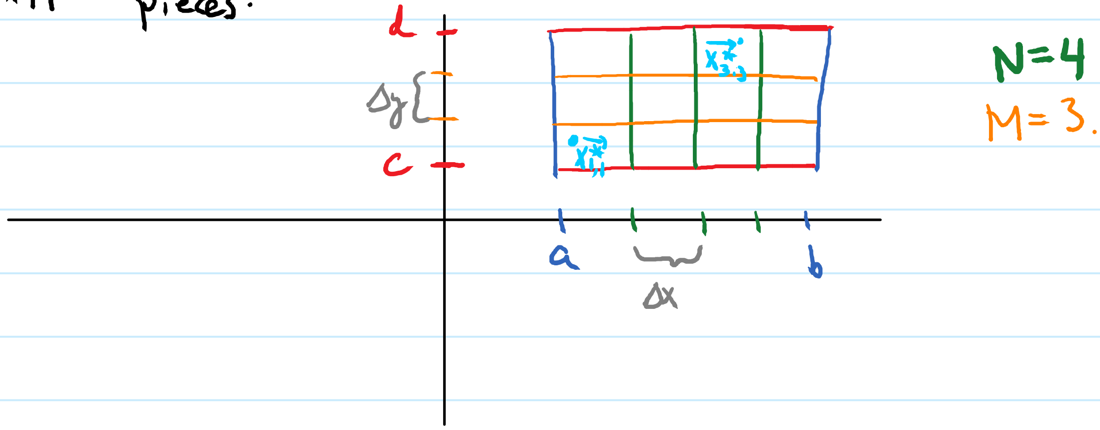
Represents the (signed) volume under the graph of $f(x,y)$.

We viewed these as iterated integrals.

Today we give a Rigorous definition via Riemann Sums:

Partition the interval $[a,b]$ into N pieces and partition $[c,d]$ into M pieces, of equal size.

this gives us a partition of the rectangle R into $N \times M$ pieces:



Let $R_{i,j}$ be the Cartesian Product of the i^{th} subinterval of $[a,b]$ with the j^{th} subinterval of $[c,d]$.

Note: R_{ij} has area $\Delta x \Delta y$

Let $\vec{x}^*_{ij}$ be an arbitrary fixed point in R_{ij}.

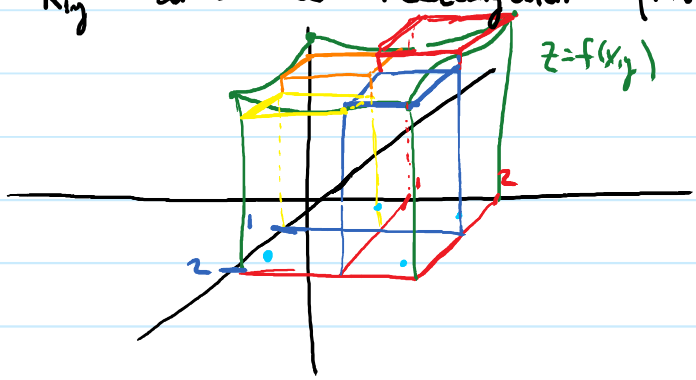
Then define

$$S_{N,M} = \sum_{j=1}^{M} \sum_{i=1}^{N} f(\vec{x}^*_{ij})\,\Delta x\,\Delta y$$

$S_{N,M}$ is called a Riemann Sum.

What is $S_{N,M}$ geometrically? For each i,j, $\Delta x \Delta y = \text{Area}(R_{ij})$.

So $f(\vec{x}_{ij}^*)\Delta x \Delta y$ is the volume of the rectangular prism with base R_{ij} and height $f(\vec{x}_{ij}^*)$. I.e., above R_{ij} draw a rectangular prism of height $f(\vec{x}_{ij}^*)$.



Then $S_{N,M}$ = Sum of volumes of rectangular prisms

is an estimate for the volume under the graph via boxes. This is analagous to estimating the area under the curve of a one variable fcn via rectangles.

As $N, M \longrightarrow \infty$, the partition gets finer, and the approximation gets better and better. Therefore, if we take limits, we get the double integral:

$$\iint_R f(x,y)\,dA \overset{\text{definition}}{=} \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{j=1}^{M}\sum_{i=1}^{N} f(\vec{x}_{ij}^*)\,\Delta x\,\Delta y$$

Remark: It is easy to see

$$\sum_{j=1}^{M}\sum_{i=1}^{N} f(x,y)\,\Delta x \Delta y = \sum_{i=1}^{N}\sum_{j=1}^{M} f(\vec{x}_{ij}^*)\,\Delta y\,\Delta x$$

I.e., in Riemann Sums, we can switch the

Order of Summation. Thus, Fubini's Thm shouldn't be surprising

The limit definition is typically very difficult and impractical to use. However, __Fubini's theorem__ also states that if f is continuous on the (closed bounded) domain D, then

$$\lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{j=1}^{M}\sum_{i=1}^{N} f(\vec{x}_{ij}^{*})\,\Delta x\,\Delta y = \underbrace{\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy}_{\text{iterated integral}}.$$

So we can just do iterated integrals instead of the limit def.

__Examples:__ ① Let $R=[0,1]\times[0,2]$. Calculate $\iint_R y e^{xy}\,dA$.

If we try to integrate w.r.t. y first, we would have to integrate by parts. That sounds awful. Let's integrate w.r.t. x first.

$$\int_0^2\int_0^1 y e^{xy}\,dx\,dy = \int_0^2 \left[\left(\frac{y}{y}e^{xy}\right)\Big|_{x=0}^{x=1}\right]dx = \int_0^2 e^{y}-e^{0}\,dy$$

$$= \int_0^2 e^{y}-1\,dy$$

$$= (e^{y}-y)\Big|_{y=0}^{y=2}$$

$$= (e^2-2)-(e^0-0)$$

$$= \boxed{e^2-3}$$

__Warning (common mistake):__ Double integrals don't generally split up under mult.

$$\boxed{\iint y e^{xy}\,dx\,dy \neq \left(\int y\,dy\right)\left(\int e^{xy}\,dx\right)}$$

② $\iint_R \frac{y^2}{1+x^2}\,dA$ where $R=[0,1]\times[-1,1]$.

$$\int_0^1\int_{-1}^1 \frac{y^2}{1+x^2}\,dy\,dx = \frac{1}{3}\int_0^1 \left(\frac{y^3}{1+x^2}\right)\Big|_{-1}^{1} dx = \frac{1}{3}\int_0^1 \frac{(1)^3-(-1)^3}{1+x^2}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{2}{1+x^2}\,dx$$

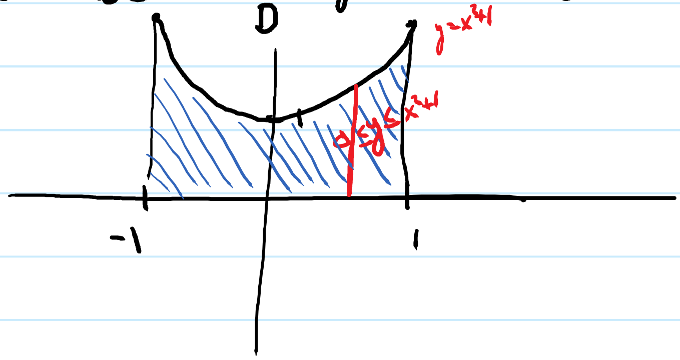
$$= \frac{2}{3}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{2}{3}\left(\tan^{-1}(x)\right)\Big|_{x=0}^{1}$$

$$\frac{2}{3}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)$$
$$= \frac{2}{3}\left(\frac{\pi}{4}-0\right)$$
$$= \frac{\pi}{6}$$

§5.3: Integrals over more general Regions!

So far, we've only integrated over rectangles. Let's suppose we want to find the volume of the region in under the paraboloid $z=1-x^2-y^2$ and above the region $D=\{(x,y) \mid {-1 \le x \le 1 \atop 0 \le y \le x^2+1}\}$. Then we want to calculate $\iint_D 1-x^2-y^2 \, dA$ where:



On D, we notice that we can find bounds for y in terms of x: $0 \le y \le x^2+1$. Since $-1 \le x \le 1$ on D, we can set up an iterated integral:

$$\underbrace{\int_{-1}^{1} \underbrace{\int_{0}^{x^2+1}}_{\text{bounds for } y}}_{\text{bounds for } x} 1-x^2-y^2 \, dy \, dx \Rightarrow \text{Inner integral} = \int_0^{x^2+1} 1-x^2-y^2 \, dy$$

$$= \left(y - x^2 y - \tfrac{1}{3}y^3\right)\Big|_{y=0}^{y=x^2+1}$$

Note: $\int^{x^2+1} f(x,y)\,dy =$ Area of slice with fixed x value

$$= (x^2+1) - x^2(x^2+1) - \tfrac{1}{3}(x^2+1)^3$$
$$= \cancel{x^2}+1 - x^4 - \cancel{x^2} - \tfrac{1}{3}(x^6+3x^4+3x^2+1)$$
$$= -\tfrac{1}{3}x^6 - 2x^4 - x^2 + \tfrac{2}{3}$$

So $\int_{-1}^{1}\int_{0}^{x^2+1} 1-x^2-y^2 \, dy \, dx = \int_{-1}^{1} -\frac{1}{3}x^6 - 2x^4 - x^2 + \frac{2}{3} \, dx$

$$= 2\int_0^1 -\frac{1}{3}x^6 - 2x^4 - x^2 + \frac{2}{3}\,dx$$

$$= 2\left(-\frac{1}{21}x^7 - \frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{2}{3}x\right)\Big|_0^1$$

$$= 2\left(-\frac{1}{21} - \frac{2}{5} - \frac{1}{3} + \frac{2}{3}\right)$$

$$= 2\left(\frac{-4}{35}\right)$$

$$= \frac{-8}{35}$$

integrating an even fcn!

Warning: ① The bounds on the "outer integral" should never have one of the variables. (We have to actually get a number at the end.) In this case, switching the order of integration takes some work. We can't just say:

$$\cancel{\int_{-1}^{1}\int_0^{x^2+1} 1-x^2-y^2\,dy\,dx = \int_0^{x^2+1}\int_{-1}^{1} 1-x^2-y^2\,dx\,dy}$$

↑ variable on the outside!

② The bounds on the integral should never involve the dummy variable. This is an easy way to check for mistakes!

$$\int_0^{x^2+1} 1-x^2-y^2\,dx$$

doesn't make sense.

Thm: If D is a region given by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

This type of region is called y-Simple.

If D is a region given by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

This type of region is called X-Simple.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 15, 2017

Section:

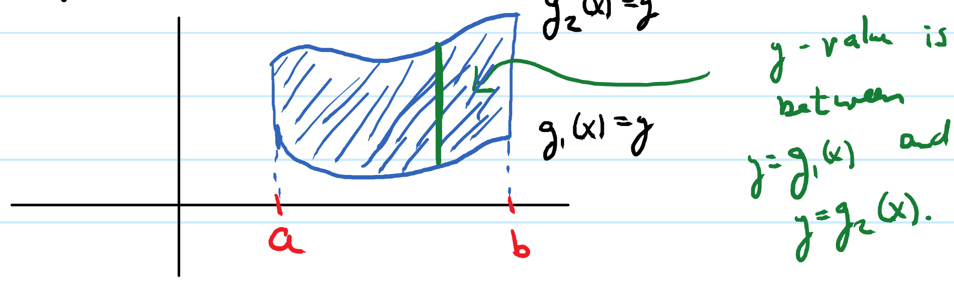
- [ ] §5.3
- [ ] §5.4

Topics Covered:

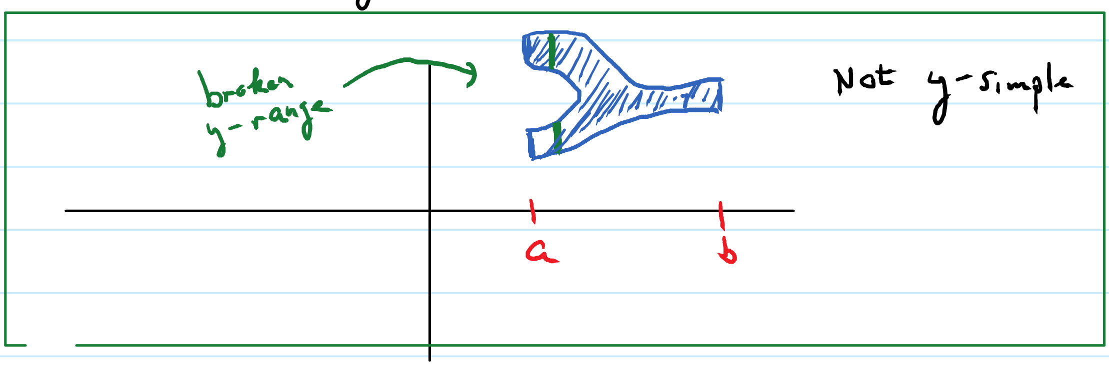
- [ ] Double integrals over more general regions
- [ ] Changing the order of integration

§5.3: Integrals over more general Regions:

Suppose we have a region, D, in the xy-plane that can be described as $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ where $g_1(x)$, $g_2(x)$ are fcns of x.

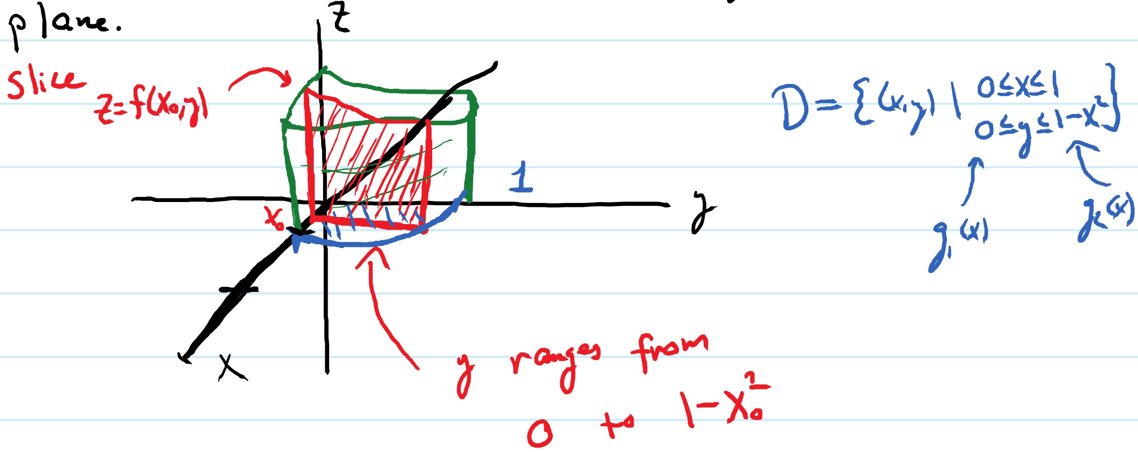


This type of region is called y-simple because the bounds for y can be written as fcns of x.



Let f be a cont. fcn on a y-simple region D. (say $f(x,y) \ge 0$). Then we can still use Cavalieri's principle to calculate the volume under $z = f(x,y)$ and above the xy-plane.

Ex:



For a fixed x-value, x_0, the area of a slice of the solid with $x = x_0$ is the area under the graph of $z = f(x_0, y)$ as $g_1(x) \le y \le g_2(x)$. If $A(x_0)$ is the area of a slice with x_0 fixed, then

$$A(x_0) = \int_{g_1(x_0)}^{g_2(x_0)} f(x_0, y)\, dy \qquad (x_0 \text{ constant!})$$

Therefore, the double integral $\iint_D f(x,y)\, dA$ can be expressed as an iterated integral by

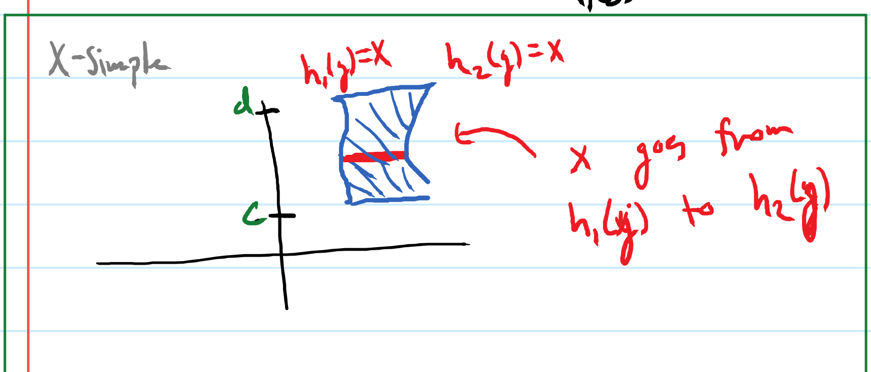
$$\iint_D f(x,y)\, dA = \int_a^b A(x)\, dx$$
$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx.$$

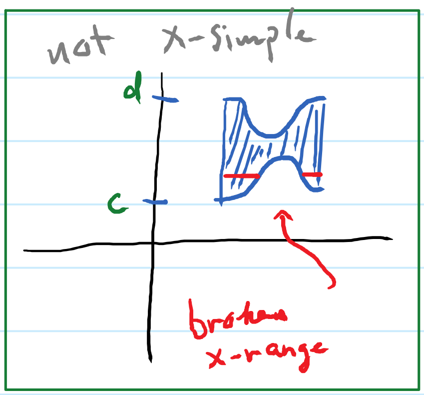
On the other hand, if D is an x-simple region. I.e., D can be described as

$$c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

(bounds for x are fcns of y). Then

$$\iint_D f(x,y)\, dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$$





Example: Let $D = \left\{ (x,y) \;\middle|\; \begin{matrix} 1 \le y \le 4 \\ \sqrt{y} \le x \le 4 \end{matrix} \right\}$

Calculate $\displaystyle\iint_D \frac{\sqrt{y}}{x^2}\, dA.$

Sol: Since D is written as an x-simple region, we should integrate w.r.t. x-first:

$$\int_1^4 \int_0^{\sqrt{y}} \frac{\sqrt{y}}{x^2}\, dx\, dy$$

Warning: ① Outer integral should not have a variable!
② In general, the bounds should never involve the dummy variable!

mistake

Common Mistake: ① $\displaystyle\iint_D \frac{\sqrt{y}}{x}\, dA = \int_{\sqrt{y}}^4 \int_1^4 \frac{\sqrt{y}}{x^2}\, dy\, dx$

② $\displaystyle\iint_D \frac{\sqrt{y}}{x}\, dA = \int_1^4 \int_{\sqrt{y}}^4 \frac{\sqrt{y}}{x}\, dy\, dx$

mistake

Inner integral (y-fixed): $\displaystyle\int_{\sqrt{y}}^4 \frac{\sqrt{y}}{x^2}\, dx$

$$= \sqrt{y} \int_{\sqrt{y}}^4 x^{-2}\, dx$$

$$= \sqrt{y} \left(-\frac{1}{x} \right) \Big|_{x=\sqrt{y}}^{x=4}$$

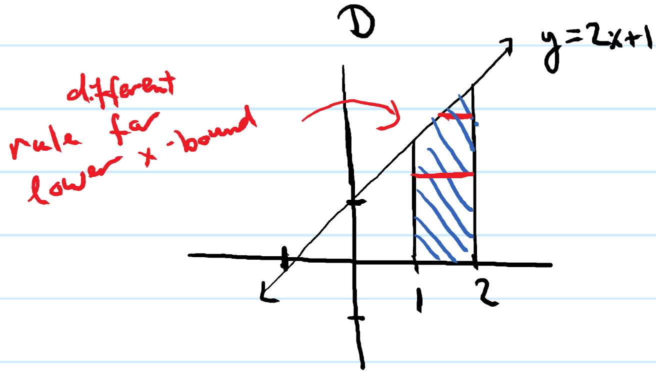
$$= \sqrt{y} \left(-\frac{1}{4} + \frac{1}{\sqrt{y}} \right)$$

$$= -\frac{\sqrt{y}}{4} + 1.$$

$$\Rightarrow \iint_D \frac{\sqrt{y}}{x}\, dA = \int_1^4 \left(1 - \frac{\sqrt{y}}{4} \right) dy = \left(y - \frac{1}{4}\left(\frac{y^{3/2}}{\left(\frac{3}{2}\right)} \right) \right) \Big|_{y=1}^{y=4} = \left(y - \frac{1}{6} y^{\frac{3}{2}} \right) \Big|_{y=1}^{y=4}$$

$$\left(4 - \tfrac{1}{6}(4)^{3/2}\right) - \left(1 - \tfrac{1}{6}(1)^{3/2}\right) = \left(3 + \tfrac{1+8}{6}\right)$$
$$= 3 + \tfrac{3}{2}$$
$$= \frac{9}{2}$$

Ex: Calculate $\iint_D e^{x^2+x}\,dA$ where D is the region bound by $x=1$, $x=2$, $y=0$, $y=2x+1$.



Note: D is not x-Simple since if choose different y-values, the "rule" for the bound changes. D is y Simple since for any x-value,

$$0 \le y \le 2x+1.$$

The overall range for x on D is $1 \le x \le 2$.
Therefore, we get the iterated integral:

$$\iint_D e^{x^2+x}\,dA = \int_1^2 \int_0^{2x+1} e^{x^2+x}\,dy\,dx$$

Note: It's a good thing we're integrating w.r.t. y first since $\int e^{x^2+x}dx$ is not possible with "elementary" techniques.

Inner Integral: (x -fixed)

$$\int_0^{2x+1} e^{x^2+x}\,dy = \left(y e^{x^2+x}\right)\Big|_{y=0}^{y=2x+1}$$
$$= \boxed{(2x+1)e^{x^2+x}}$$

Plugging back into the outer fcn, we get

$$\int_1^2 (2x+1)e^{x^2+x}\,dx \quad . \quad \text{Let } u = x^2+x \Rightarrow du = (2x+1)dx$$

now we can substitute!

$$\Rightarrow \int_1^2 (2x+1)e^{x^2+x}\,dx$$
$$= \int_{u(1)}^{u(2)} e^u\,du$$
$$= \left(e^u\right)\Big|_{u(1)}^{u(2)}$$
$$= \left(e^{x^2+x}\right)\Big|_1^2 = e^6 - e^2$$

In this example, we saw that we could not integrate w.r.t. x in the original fcn. However, once we integrated w.r.t y, the integral became possible. This example illustrates why we might want to change the order of integration.

We will see how to do this next time.

Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 17, 2017

Section:

- §5.4

Topics Covered:

- Changing the order of integration

§5.4: Changing the Order of Integration:

Last time, we saw that sometimes it may be beneficial to change the order of integration in order to have a reasonable integrand. Let's formalize this idea.

Let D be a region in the xy-plane that is both x-simple and y-simple. I.e., D can be written in the form

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\} \longleftrightarrow \text{(x-simple)}$$

and

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \longleftrightarrow \text{(y-simple)}$$

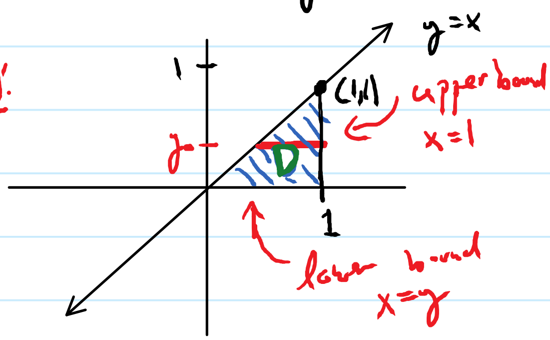
Rmk: We call such a domain simple.

If $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous, then we can write $\iint_D f\,dA$ as an iterated integral in two different ways:

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Ex: Let D be the domain bounded by $x=0$, $x=1$, $y=0$, $y=x$. If $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous, express $\iint_D f\,dA$ as an iterated integral in two different ways.

Sol:

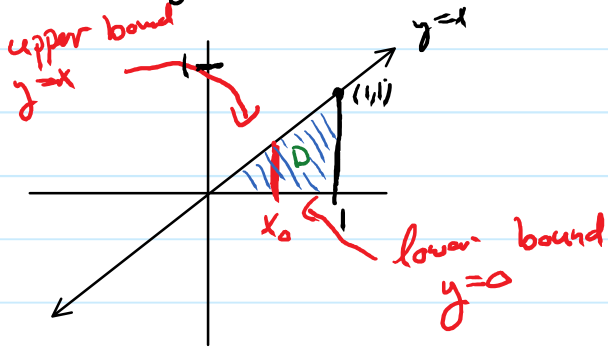


As an x-simple domain, if we fix a y-value, x is bounded from below by $x=y$, and x is bounded from above by $x=1$.

meanwhile, the largest y-value on D is 1, and the smallest y-value on D is 0. So as ~~an~~ x-simple domain,

$$D=\left\{(x,y)\,\middle|\,\begin{matrix}0\le y\le 1\\ y\le x\le 1\end{matrix}\right\}. \Rightarrow \boxed{\iint_D f\,dA=\int_0^1\int_y^1 f(x,y)\,dx\,dy.}$$

As a y-simple domain, if we fix an x-value, then y ranges from $y=0$ to $y=x$. Meanwhile, the smallest x-value on D is 0 while the largest is 1. So as a y-simple domain,



$$D=\left\{(x,y)\,\middle|\,\begin{matrix}0\le x\le 1\\ 0\le y\le x\end{matrix}\right\} \Rightarrow$$

$$\boxed{\iint_D f\,dA=\int_0^1\int_0^x f(x,y)\,dy\,dx}$$

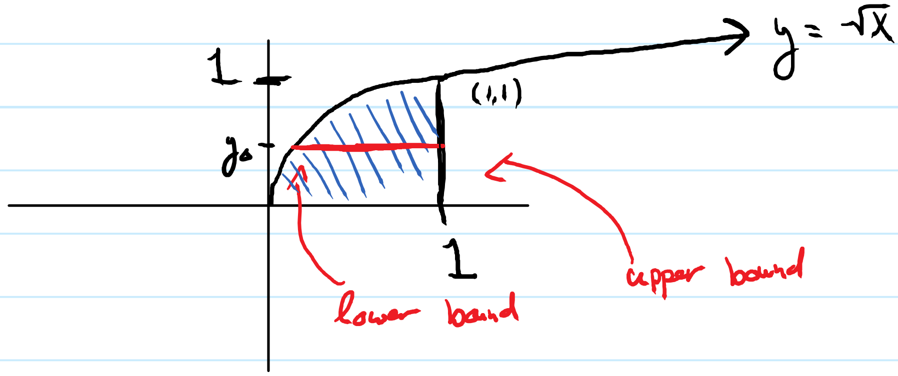
Warning: A common mistake is to simply replace every x with a y in the bounds (or vice versa).

$$\int_0^1\int_0^x f(x,y)\,dy\,dx \neq \int_0^1\int_0^y f(x,y)\,dx\,dy$$

Ex: Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{x}} x\,e^{y^5-5y}\,dy\,dx$

Sol: Unfortunately, we don't have an "elementary" way to evaluate the integral $\int_0^{\sqrt{x}} x\,e^{y^5-5y}\,dy$. Let's switch the order of integration.

Step 1: (Draw the domain): We are integrating over the region $D\subseteq\mathbb{R}^2$ defined by: $0\le x\le 1$
$0\le y\le\sqrt{x}$



To express D as an X-simple domain, we fix a y value and find the range for x in terms of y. For a fixed y-value, the smallest x can be is when x is on the curve $y=\sqrt{x} \Rightarrow \boxed{x=y^2}$. The largest x value is 1. Meanwhile, the absolute bounds for y on D are $0 \le y \le 1$. Therefore, we have

$0 \le y \le 1$, $y^2 \le x \le 1$. By switching the order of integration, we get

$$\int_0^1\int_0^{\sqrt{x}} x e^{y^5-5y}\,dy\,dx = \int_0^1\int_{y^2}^1 x e^{y^5-5y}\,dx\,dy$$

Now we can evaluate the inner integral:

$$\int_{y^2}^1 x\, e^{y^5-5y}\,dx \quad \left(\frac{x^2}{2} e^{y^5-5y}\right)\Big|_{x=y^2}^{x=1} = \left(\frac{1}{2}-\frac{(y^2)^2}{2}\right)e^{y^5-5y}$$
$$= \frac{1}{2}(1-y^4)\,e^{y^5-5y}$$

$$\Rightarrow \int_0^1\int_{y^2}^1 x e^{y^5-5y}\,dx\,dy = \int_0^1 \frac{1}{2}(1-y^4)\,e^{y^5-5y}\,dy$$

Now we make the substitution, $u = y^5-5y$ $(\Rightarrow u(0)=0,\ u(1)=-4$

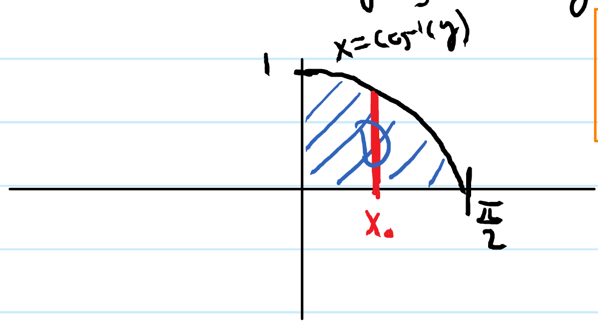
$$\Rightarrow du = (5y^4-5)\,dy$$
$$\Rightarrow -\frac{1}{5}du = (1-y^4)\,dy$$

$$\Rightarrow \int_0^1 \frac{1}{2}(1-y^4)e^{y^5-5y}\,dy = -\frac{1}{10}\int_0^{-4} e^u\,du = -\frac{1}{10}(e^{-4}-e^0)$$
$$= -\frac{1}{10}(e^{-4}-1)$$

Ex: Evaluate $\int_0^1 \int_0^{\cos^{-1}(y)} \sqrt{1+\sin(x)}\, dx\, dy$

Since $\int \sqrt{1+\sin(x)}\, dx$ isn't obvious, let's change the order of integration. Here, the domain is written as:

$0 \le x \le \cos^{-1}(y)$, $0 \le y \le 1$



$x=\cos^{-1}(y) \Leftrightarrow y=\cos(x)$ and $0 \le x \le \pi$

By definition of $f(y)=\cos^{-1}(y)$!

We need to fix an x-value and solve for y in terms of x. Since $x=\cos^{-1}(y) \Leftrightarrow y=\cos(x)$ & $x \in [0,\pi]$, the upper bound for y in terms of x is $y=\cos(x)$, and the lower bound is $y=0$. $\Rightarrow$ $\boxed{0 \le y \le \cos(x).}$

Since $x=\cos^{-1}(y)$, and $y=0$ intersect when $x=\frac{\pi}{2}$, the absolute bounds for x are $\boxed{0 \le x \le \frac{\pi}{2}.}$

Therefore, $\int_0^1 \int_0^{\cos^{-1}(y)} \sqrt{1+\sin(x)}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{\cos(x)} \sqrt{1+\sin(x)}\, dy\, dx$

$$= \int_0^{\frac{\pi}{2}} \left(y\sqrt{1+\sin(x)} \right)\Big|_{y=0}^{y=\cos(x)} dx$$

$$= \int_0^{\pi/2} \cos(x)\sqrt{1+\sin(x)}\, dx.$$

Now we make the substitution

$u = 1+\sin(x) \Rightarrow$ • $u(0)=1$

• $u(\frac{\pi}{2})=2$

• $du=\cos(x)dx$

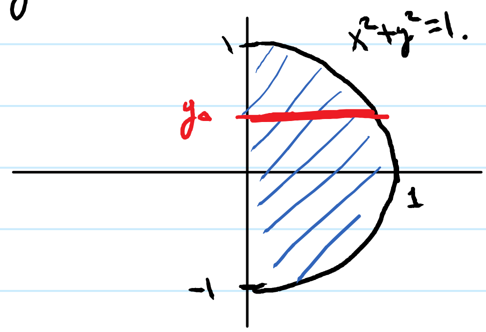
- $du = \cos(x)dx$

$$\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\sin(x)}\,dx = \int_1^2 \sqrt{u}\,du$$
$$= \left(\frac{2}{3}u^{3/2}\right)\Bigg|_{u=1}^{u=2}$$
$$= \boxed{\frac{2}{3}\left(2^{3/2}-1\right)}$$

<u>Example:</u> Evaluate $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{\sqrt{(x^2+y^2)}}\,dy\,dx.$

<u>Sol:</u> Let D be the domain of integration. Then D can be described by $\{(x,y) \mid -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le x \le 1\}$.

Note: the fcns $y=\sqrt{1-x^2}$, $y=-\sqrt{1-x^2}$ describe the upper and lower hemispheres of the unit circle. Since $0 \le x \le 1$, D is the right half of the unit disc.



To switch the order of integration, we need to fix a y-value and express the bounds of x in terms of y. If y is fixed, the lower bound for x is 0 and the upper bound for x is when the point hits the unit circle $x^2+y^2=1 \Rightarrow x=\sqrt{1-y^2}$ (only positive since we already knew $x \ge 0$.)

Meanwhile the absolute bounds for y on D are

$$D=\left\{(x,y) \,\middle|\, \begin{array}{c} -1 \le y \le 1 \\ 0 \le x \le \sqrt{1-y^2} \end{array}\right\}$$

$$\Rightarrow \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{(x^2+y^2)^{1/2}}\,dy\,dx = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\,dx\,dy.$$

Inner Integral: $\int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\,dx.$

(y is const.)

Let $u = x^2+y^2 \Rightarrow$

- $u(0) = y^2$
- $u(\sqrt{1-y^2}) = (\sqrt{1-y^2})^2 + y^2 = 1$
- $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx.$

$$\int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\,dx = \frac{1}{2}\int_{y^2}^{1} \frac{1}{u^{1/2}}\,dx = \frac{1}{2}\int_{y^2}^{1} u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left(2u^{\frac{1}{2}}\right)\Big|_{u=y^2}^{u=1}$$

$$= \frac{1}{2}\left(2 + 2(y^2)^{1/2}\right)$$

$$= 1+y.$$

Outer integral: $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{-1}^{1} 1+y\,dy = \left(y+\frac{1}{2}y^2\right)\Big|_{-1}^{1}$

$$= \left(1+\frac{1}{2}\right) - \left(-1+\frac{1}{2}\right)$$

$$= \boxed{2}$$